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Existence of maximal and minimal solutions initial value problem for the system of fractal differential equations

Mohammad Sajid[1], Hemanta Kalita[2*], Gülizar Gülenay Zengin[3] and Lucas Wangwe[4]

*Correspondence:
hemanta30kalita@gmail.com
[2]Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University, Bhopal-Indore Highway, India
Full list of author information is available at the end of the article

Abstract

Differential equation refers to an equation that includes a function and its derivatives. These equations serve to model real-world situations where rates of change are significant. They are classified as either ordinary differential equations (ODEs) or partial differential equations (PDEs), depending on whether the unknown function is dependent on one or several independent variables, respectively. This paper presents a thorough investigation into fractal differential inequalities linked with an initial value fractal differential equation. It establishes the existence of a solution to this equation and demonstrates the convergence of both minimal and maximal solutions. Additionally, the paper introduces a comparative principle for evaluating solutions to the initial value problem associated with the fractal differential equation, ensuring a detailed and rigorous analysis of this subject.

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1 Introduction

A fractal refers to a geometric shape or pattern characterized by self-similarity, which implies that its structure is replicated at multiple scales. Examples of this include the Mandelbrot set, the Koch snowflake, and tree-like branching structures. Fractals, recognized for their noninteger dimensions, represent a notable departure from standard geometric shapes. Unlike linear paths, perfect spheres, or conical structures found in classical geometry, natural phenomena such as lightning, clouds, and mountains showcase intricate forms that elude conventional mathematical description. Thus, the study of fractal geometry has become indispensable for capturing the complexity inherent in these natural shapes [5, 11]. Fractal calculus is an advanced form of conventional calculus that deals with functions associated with fractal supports and situated in fractal spaces. Fractal calculus is a sophisticated mathematical framework that broadens the scope of traditional calculus, permitting the exploration of equations with solutions that manifest as functions with fractal attributes, including fractal sets and curves. Its charm lies in its refined elegance and algorithmic strategies, which are notably superior to other methodologies [13]. A fractal spiral is defined as a spiral that possesses fractal characteristics, where its details recur at progressively smaller scales, resulting in a complex and intricate pattern. The fractal characteristics of elliptical polynomial spirals, which are planar spirals exhibiting varying polynomial decay rates along the two axes, were examined in [2].

A variety of challenges in applied mathematics can be framed as initial value problems involving ordinary differential equations or collections of such equations. The numerical resolution of initial value problems in ordinary differential equations utilizing boundary value methods is discussed in [1]. One can find details of differential equations for fractional derivatives in [12, 14]. The significant adoption of fractional differential equations in areas like fluid dynamics, finance, and other sophisticated systems has incited a detailed study of the properties of fractional derivatives and their geometric and physical implications but fractional and fractal derives are not alike [4]. The solvability of integral equations associated with initial value problems for a nonlinear differential equation of fractional order has been discussed by several researchers [9, 10]. New insights into the existence of solutions for first-order ordinary differential equations with deviated arguments were explored in [6]. Their overarching framework includes delay differential equations, which allow for deviations that may depend on the unknown solutions. The existence results are founded on innovative definitions of lower and upper solutions. Furthermore, the discussion extends to minimal and maximal solutions in the standard set-theoretic framework [6]. Recently, the existence and asymptotic estimates of both the maximal and minimal solutions for a coupled tempered fractional differential system with varying orders were examined [3]. Fractal retarded, neutral, and renewal delay differential equations with constant coefficients were addressed using the method of steps in conjunction with the Laplace transform [8]. Recently fractal integral equation was introduced in [7].

We were motivated by the work of [12] to introduce initial value problem for fractal differential equation, and [6] encouraged us to find the minimal and maximal solutions of the initial value fractal differential equation.

The structure of the article is as follows: In Sect. 1, we recall several definitions and existing results that are useful for our next section. In Sect. 2, theory of fractal differential inequalities is presented. In Sect. 3, the existence and fractal continuation of solutions to an initial value problem for fractal differential equations that involve fractal differential inequalities are studied. In the last section of the article, the minimal and maximal solutions of our assumed initial value fractal differential equation convergences in the sense of Picard's approximation method for fractal differential equation through fractal spiral are shown. Lastly, we explore various comparison theorems for fractal differential equations, along with an estimation of a function that meets a fractal differential inequality.

2 Preliminaries

Let $\mathcal{F} \subset \mathbb{R}^n$ be a fractal curve. Throughout the article $\mathbb{R}$ will be a set of real numbers. We denote fractal spiral by R_0, which is approximated by rectangles.

Definition 2.1 [13] A fractal curve $\mathcal{F} \subset \mathbb{R}^n$ is classified as continuously parametrizable if there is a continuous function $\Im : [\tau_1, \kappa_1] \to \mathcal{F}$ such that $[\tau_1, \kappa_1] \subset \mathbb{R}$ and the function is both one-to-one and onto on $\mathcal{F}$.

Let $P = \left\{\tau = \ell_0 < \ell_1 < \ell_2, \dots, \ell_p = \kappa\right\} = \left\{(\ell_\iota, [\tau_\iota, \kappa_\iota])\right\}_{\iota=1}^{p}$ be any tagged partition of $[\tau, \kappa]$. The flag function pertaining to $\mathcal{F}$ is

$$\mathfrak{F}(\mathcal{F}, [\tau, \kappa]) = \begin{cases} 1, \text{whenever } \mathcal{F} \cap [\tau, \kappa] \neq \emptyset; \\ 0, \text{under other conditions.} \end{cases}$$

We recall fractal summation of $\mathcal{F}$ as follows:

$$\sigma^{\alpha}[\mathcal{F}, [\tau, \kappa]] = \sum_{\iota=1}^{p} \Gamma(\alpha + 1)|\ell_\iota - \ell_{\iota-1}|^{\alpha} \theta(\mathcal{F}, [\tau_\iota, \kappa_\iota]),$$

where $\ell_{\iota-1} < \ell_\iota$ and $0 < \alpha \leq 1$. We denote by $|.|$ the Euclidean norm in $\mathbb{R}^n$. This expression analyzes a measure related to the partition P and the fractal curve $\mathcal{F}$ in a particular way through the parameter α.

Definition 2.2 Let $\delta > 0$ be given and P be any tagged partition of $[\tau, \kappa]$. A coarse-grained mass $M_{\delta}^{\alpha}(\mathcal{F}, \tau, \kappa)$ is defined as follows: $M_{\delta}^{\alpha}(\mathcal{F}, \tau, \kappa) = \inf_{\{P: \ |P| \leq \delta\}} \sigma^{\alpha}$, where $|P| = \max_{1 \leq \iota \leq m} (\ell_\iota - \ell_{\iota-1})$.

Definition 2.3 We define the mass function $M^{\alpha}(\mathcal{F}, \tau, \kappa)$ as the limit of the coarse-grained mass $M_{\delta}^{\alpha}(\mathcal{F}, \tau, \kappa)$ as δ gets closer to zero. That is, $M^{\alpha}(\mathcal{F}, \tau, \kappa) = \lim_{\delta \to 0} M_{\delta}^{\alpha}(\mathcal{F}, \tau, \kappa)$.

Definition 2.4 The fractal dimension of the curve $\mathcal{F}$, denoted by $dim_{\gamma}(\mathcal{F})$, is defined as follows:

$$\begin{aligned} dim_{\gamma}(\mathcal{F}) &= \inf\left\{\alpha : \ M^{\alpha}(\mathcal{F}, \tau, \kappa) = 0\right\} \\ &= \sup\left\{\alpha : \ M^{\alpha}(\mathcal{F}, \tau, \kappa) = \infty\right\}. \end{aligned}$$

The fractal dimension of $\mathcal{F}$ is characterized by the highest value of α at which the mass function $M^{\alpha}(\mathcal{F}, \tau, \kappa)$ approaches infinity and the lowest value of α at which it approaches zero. This fractal dimension serves as a measure of complexity and self-similarity of the curve $\mathcal{F}$, reflecting how effectively the curve occupies space.

Definition 2.5 The staircase function $\mathcal{S}_{\mathcal{F}}^{\alpha} : [\tau_1, \kappa_1] \to \mathbb{R}$ associated with the fractal curve $\mathcal{F}$ is defined as follows:

$$\mathcal{S}_{\mathcal{F}}^{\alpha}(\ell) = \begin{cases} M^{\alpha}(\mathcal{F}, \tau_0, \ell), \text{ if } \ell \geq \tau_0; \\ -M^{\alpha}(\mathcal{F}, \ell, \tau_0), \text{ if } \ell < \tau_0, \end{cases}$$

where $\ell \in [\tau_1, \kappa_1]$, which is a subset of $[\tau, \kappa]$. The expression $\mathcal{S}_{\mathcal{F}}^{\alpha}(\ell) = \mathcal{S}(\varpi)$ corresponds to the value of $\mathcal{S}(\varpi)$, which denotes the fractal dimension of the partition of $\mathcal{F}$ from the initial point τ_0 to ϖ. The function $\Im(\ell)$ maps the value "ℓ" to a specific point ϖ on the fractal curve $\mathcal{F}$.

The n-dimensional real-valued Euclidean space of elements $\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n)$ is represented by $\mathbb{R}^n$. As an element of $\mathbb{R}^{n+1}$, we indicate $(\ell, \mathfrak{u})$. Consider any convenient norm $||\mathfrak{u}||$. Assume that the fractal continuous $\mathcal{F}$-curve from every fractal subset $\mathcal{E}$ into $\mathbb{R}^n$ is

$\mathcal{C}(\mathcal{E},\mathbb{R}^n)$. $\Im \in \mathcal{C}(\mathcal{E},\mathbb{R}^n)$ is what we write if $\Im$ belongs to this class. We adopt the following $\mathcal{F}$-derivatives for any function $\Im \in \mathcal{C}((\ell_0,\ell_0+\tau)),\mathbb{R}^n)$.

Definition 2.6 [13]

1. The left $\mathcal{F}$-derivative of a fractal function $\Im : \mathcal{F} \to \mathbb{R}^n$ at any specified point $\ell_0 \in \mathcal{F}$ is established as

$$\mathcal{D}^{\alpha}_{\mathcal{F}}\Im_{-}(\ell_0) = \mathcal{F}\text{-}\lim_{\ell\to\ell_0} \frac{\Im(\ell)-\Im(\ell_0)}{\mathcal{S}(\ell)-\mathcal{S}(\ell_0)},$$

   where $\mathcal{S}(\ell)$ is the staircase function.
2. The right $\mathcal{F}$-derivative of a fractal function $\Im : \mathcal{F} \to \mathbb{R}^n$ at any specified point $\ell_0 \in \mathcal{F}$ is established as

$$\mathcal{D}^{\alpha}_{\mathcal{F}}\Im_{+}(\ell_0) = \mathcal{F}\text{-}\lim_{\ell\to\ell_0} \frac{\Im(\ell)-\Im(\ell_0)}{\mathcal{S}(\ell)-\mathcal{S}(\ell_0)},$$

   where $\mathcal{S}(\ell)$ is the staircase function.

In the left $\mathcal{F}$-derivative, $\ell \to \ell_0$ means ℓ approaches ℓ_0 from the left side of ℓ_0. In the right $\mathcal{F}$-derivative, $\ell \to \ell_0$ means ℓ approaches ℓ_0 from the right side of ℓ_0.

Theorem 2.7 [13, *Theorem* 50] *If* $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell)$ *exists for all* $\ell \in (\ell_0,\ell_0+\tau)$, *then* $\Im(\ell)$ *is* $\mathcal{F}$-*continuous in* $(\ell_0,\ell_0+\tau)$.

Consider $\mathfrak{S}$ to be an at-most countable subset of the interval $[\ell_0,\ell_0+\tau)$.

Lemma 2.8 *For* $\ell \in [\ell_0,\ell_0+\tau)-\mathfrak{S}$, *if* $\Im \in \mathcal{C}([\ell_0,\ell_0+\tau),\mathbb{R})$ *and* $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell) \le 0$, *then* $\Im(\ell)$ *is decreasing in* ℓ *on* $[\ell_0,\ell_0+\tau]$.

Proof Let $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell) = \mathcal{F}\text{-}\lim_{\ell\to\ell_0} \frac{\Im(\ell)-\Im(\ell_0)}{\mathcal{S}(\ell)-\mathcal{S}(\ell_0)}$ exist and be negative, i.e., $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell) < 0 \; \forall \; \ell \in [\ell_0,\ell_0+\tau)-\mathfrak{S}$. If possible, suppose that $\Im$ fails to be nonincreasing on $[\ell_0,\ell_0+\tau)$. It can be stated that there are ℓ_1 and ℓ_2 for which $\ell_0 \le \ell_1 \le \ell_2 \le \ell_0+\tau$, satisfying the inequality $\Im(\ell_1) > \Im(\ell_2)$. Let $y \in \big(\Im(\ell_2),\Im(\ell_1)\big)$. Considering that $\Im$ exhibits $\mathcal{F}$-continuity, according to the intermediate value theorem, there exists a value z within the interval $[\ell_1,\ell_2]$ such that $\Im(z)=y$. Clearly, the intersection of the set $\big\{\ell : \Im(\ell)=y\big\}$ and the interval $[\ell_1,\ell_2]$ contains elements. Suppose $p_0 = \sup\big\{\ell : \Im(\ell)=y\big\} \cap [\ell_1,\ell_2]\big\}$. As it stands, $\Im(\ell_2)$ is smaller than y, and since $\Im$ exhibits $\mathcal{F}$-continuity, this implies that p_0 is less than ℓ_2. Thus $\Im(\ell) < y \; \forall \; \ell \in (\ell_0,\ell_2]$ and the set $\big\{\ell : \Im(\ell)=y\big\}$ is closed as $\Im$ is $\mathcal{F}$-continuous. So, $\Im(p_0)=y$, and this implies $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im_{+}(p_0) \ge 0$, a contradiction. Hence $\Im(\ell)$ is nonincreasing in ℓ on $[\ell_0,\ell_0+\tau)$. □

Definition 2.9 [6] Let $\Im_M(\ell)$ represent a solution to any initial value problem defined on the interval $[\ell_0,\ell_0+\tau)$. The function $\Im_M(\ell)$ is classified as a maximal solution of the specified initial value problem if, for every solution existing on $[\ell_0,\ell_0+\tau)$, the condition $\Im(\ell) \le \Im_M(\ell)$ for all ℓ in $[\ell_0,\ell_0+\tau)$ is satisfied.

Definition 2.10 [6] Let $\Im_m(\ell)$ represent a solution to any initial value problem defined on the interval $[\ell_0,\ell_0+\tau)$. The function $\Im_m(\ell)$ is classified as a minimal solution of the specified initial value problem if, for every solution of the problem that exists on $[\ell_0,\ell_0+\tau)$, the inequality $\Im(\ell) \ge \Im_m(\ell)$ holds for all ℓ in the interval $[\ell_0,\ell_0+\tau)$.

3 Fractal differential inequalities

In this segment of the article, we present the concept of fractal differential inequalities. We shall analyze a system of first-order fractal differential equations accompanied by the initial condition

$$\mathcal{D}_{\mathcal{F}}^{\alpha}\Im = \hbar(\ell, \Im), \ \Im(\ell_0) = \Im_0, \tag{1}$$

where $\Im_0 = (\Im_{10}, \Im_{20}, \ldots, \Im_{n0})$ and $\hbar \in \mathcal{C}(\mathcal{E}, \mathbb{R}^n)$.

The solution to the initial value problem (1) is characterized as a fractal differentiable function of ℓ, satisfying $\Im(\ell_0) = \Im_0(\ell, \Im(\ell)) \in \mathcal{E}$ and $\mathcal{D}_{\mathcal{F}}^{\alpha}\Im(\ell) = \hbar(\ell, \Im(\ell))$ over an ℓ-fractal interval J containing ℓ_0. The function $\Im(\ell)$ possesses the properties of fractal continuity as well as fractal differentiability. Based on these requirements for the fractal continuous function $\Im(\ell)$, it can be concluded that

$$\Im(\ell) = \Im_0 + \int_{\ell_o}^{\ell} \hbar(s, \Im(s) d_{\mathcal{F}}^{\alpha} s, \ \ell \in J. \tag{2}$$

Let $\mathcal{E}$ represent an α-open $(\ell, \Im)$ set in the Euclidean space $\mathbb{R}^2$, with $\hbar$ being an element of $\mathcal{C}(\mathcal{E}, \mathbb{R})$. Let us investigate the scalar fractal differential equation together with the initial condition

$$\mathcal{D}_{\mathcal{F}}^{\alpha}\Im(\ell) = \hbar(\ell, \Im), \ \Im(\ell_0) = \ell_0. \tag{3}$$

Definition 3.1 Suppose that $\mathfrak{v} \in \mathcal{C}([\ell_0, \ell_0 + \tau), \mathbb{R})$ and $\mathcal{D}_{\mathcal{F}}^{\alpha}\mathfrak{v}_+(t)$ exists for $\ell \in [\ell_0, \ell_0 + \tau)$ and $(\ell, \mathfrak{v}(\ell)) \in \mathcal{E}$. We called $\mathfrak{v}(\ell)$ to be upper function if $\mathfrak{v}(\ell)$ satisfies the fractal differential inequality $\mathcal{D}_{\mathcal{F}}^{\alpha}\mathfrak{v}_+(\ell) < \hbar(\ell, \mathfrak{v}(\ell))$, $\ell \in [\ell_0, \ell_0 + \tau)$ concerning the initial value problem (3).

Definition 3.2 Suppose $\mathfrak{v} \in \mathcal{C}([\ell_0, \ell_0 + \tau), \mathbb{R})$ and $\mathcal{D}_{\mathcal{F}}^{\alpha}\mathfrak{v}_+(\ell)$ exists for $\ell \in [\ell_0, \ell_0 + \tau)$ and $(\ell, \mathfrak{v}(\ell)) \in \mathcal{E}$. We called $\mathfrak{v}(\ell)$ to be over function if $\mathfrak{v}(\ell)$ satisfies the fractal differential inequality $\mathcal{D}_{\mathcal{F}}^{\alpha}\mathfrak{v}_+(\ell) > \hbar(\ell, \mathfrak{v}(\ell))$, $\ell \in [\ell_0, \ell_0 + \tau)$ concerning the initial value problem (3).

A fundamental result can be formulated as follows.

Theorem 3.3 *Let $\mathcal{E}$ denote an α-open set that is classified as an $(\ell, \Im)$-set within $\mathbb{R}^2$, and let $\hbar$ be a function belonging to $\mathcal{C}(\mathcal{E}, \mathbb{R})$. Let us consider $\mathfrak{v}, \mathfrak{w} \in \mathcal{C}([\ell_0, \ell_0 + \tau), \mathbb{R})$ and $(\ell, \mathfrak{v}(\ell))$, $(\ell, \mathfrak{w}(\ell)) \in \mathcal{E}$, $\ell \in [\ell_0, \ell_0 + \tau)$ be given. Assume that $\mathfrak{v}(\ell_0) < \mathfrak{w}(\ell_0)$, and for $\ell \in (\ell_0, \ell_0 + \tau)$ and elsewhere, the subsequent inequalities*

$$\mathcal{D}_{\mathcal{F}}^{\alpha}\mathfrak{v}_-(\ell) \leq \hbar(\ell, \mathfrak{v}(\ell)); \ \mathcal{D}_{\mathcal{F}}^{\alpha}\mathfrak{w}_-(\ell) > \hbar(\ell, \mathfrak{w}(\ell)) \ \textit{hold}.$$

Then $\mathfrak{v}(\ell) < \mathfrak{w}(\ell)$, $\ell \in [\ell_0, \ell_0 + \tau)$.

Proof Let us prove with a contradiction approach. If possible, let $\mathfrak{v}(\ell) \not< \mathfrak{w}(\ell)$ for $\ell \in [\ell_0, \ell_0 + \tau)$, and $\left\{\ell \in [\ell_0, \ell_0 + \tau) : \ \mathfrak{w}(\ell) \leq \mathfrak{v}(\ell)\right\}$ is nonempty.

Let $\ell_1 = \inf \left\{\ell \in [\ell_0, \ell_0 + \tau) : \ \mathfrak{w}(\ell) \leq \mathfrak{v}(\ell)\right\}$. Since $\mathfrak{v}(\ell_0) < \mathfrak{w}(\ell_0)$, so $\ell_0 < \ell_1$. Furthermore, $\mathfrak{v}(\ell_1) = \mathfrak{w}(\ell_1)$ and $\mathfrak{v}(\ell) < \mathfrak{w}(\ell)$, $\ell \in [\ell_0, \ell_1)$. So, for any $h < 0$, we have

$$\frac{\mathfrak{v}(\ell_1 + h) - \mathfrak{v}(\ell_1)}{\mathcal{S}(\ell_1 + h) - \mathcal{S}(\ell_1)} > \frac{\mathfrak{w}(\ell_1 + h) - \mathfrak{w}(\ell_1)}{\mathcal{S}(\ell_1 + h) - \mathcal{S}(\ell_1)}.$$

This implies $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{-}(\ell_1) \geq \mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{w}_{-}(\ell_1)$. Next $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{-}(\ell) \leq \hbar(\ell,\mathfrak{v}(\ell))$; $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{w}_{-}(\ell) > \hbar(\ell,\mathfrak{w}(\ell))$ and $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{-}(\ell_1) \geq \mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{w}_{-}(\ell_1)$ together in $\mathfrak{v}(\ell_1) = \mathfrak{w}(\ell_1)$. Consequently, we get a contradiction $\hbar(\ell_1,\mathfrak{v}(\ell_1)) > \hbar(\ell_1,\mathfrak{w}(\ell_1))$. Hence $\left\{\ell \in [\ell_0,\ell_0+\tau) : \mathfrak{w}(\ell) \leq \mathfrak{v}(\ell)\right\}$ is empty, a contradiction, and hence $\mathfrak{v}(\ell) < \mathfrak{w}(\ell)$, $\ell \in [\ell_0,\ell_0+\tau)$. □

Lemma 3.4 *Let* $\mathfrak{v}$, $\mathfrak{w} \in \mathcal{C}([\ell_0,\ell_0+\tau),\mathbb{R})$ *and for some* $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell) \leq \mathfrak{w}(\ell)$ *for* $\ell \in [\ell_0,\ell_0+\tau) - \mathfrak{S}$. *Then* $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{-}(\ell) \leq \mathfrak{w}(\ell)$ *for* $\ell \in [\ell_0,\ell_0+\tau)$.

Proof Let $m(\ell) = \mathfrak{v}(\ell) - \int_{\ell_0}^{\ell} \mathfrak{w}(s) d^{\alpha}_{\mathcal{F}}s$. From our hypothesis, $\mathcal{D}^{\alpha}_{\mathcal{F}}m(\ell) = \mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}(\ell) - \mathfrak{w}(\ell) \leq 0$, $\ell \in [\ell_0,\ell_0+\tau) - \mathfrak{S}$. By Lemma 2.8, $m(\ell)$ is nonincreasing in ℓ on $[\ell_0,\ell_0+\tau)$. Hence $\mathcal{D}^{\alpha}_{\mathcal{F}}m_{-}(\ell) = \mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{-}(\ell) - \mathfrak{w}(\ell) \leq 0$, $\ell \in [\ell_0,\ell_0+\tau)$. □

Remark 3.5 With Lemma 3.4, it is clear that Theorem 3.3 will remain true when $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{-}(\ell) \leq \hbar(\ell,\mathfrak{w}(\ell))$ and $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{w}_{-}(\ell) \geq \hbar(\ell,\mathfrak{w}(\ell))$ hold for $\ell \in [\ell_0,\ell_0+\tau) - \mathfrak{S}$.

The subsequent theorem demonstrates that any solution to the initial value problem (1) can be confined between its upper and over functions.

Theorem 3.6 *Let* $\mathfrak{v}(\ell)$ *and* $\mathfrak{w}(\ell)$ *represent the upper and over functions, respectively, concerning the initial value problem* (1) *on the interval* $[\ell_0,\ell_0+\tau)$. *If* $\Im(\ell)$ *represents any solution to equation* (1) *satisfying on* $[\ell_0,\ell_0+\tau)$ *such that* $\mathfrak{v}(\ell_0) = \Im_0 = \mathfrak{w}(\ell_0)$, *then* $\mathfrak{v}(\ell) < \Im(\ell) < \mathfrak{w}(\ell)$, $\ell \in [\ell_0,\ell_0+\tau)$.

Proof Define $\mathfrak{w}(\ell)$ as an over function and $\Im(\ell)$ as the corresponding solution of (1). Consider $m(\ell) = \mathfrak{w}(\ell) - \Im(\ell)$. As a result, it can be stated that $\mathcal{D}^{\alpha}_{\mathcal{F}}m_{+}(\ell) > 0$. It follows that $m(\ell)$ is increasing to the right of ℓ_0 in a sufficiently small interval $\ell_0 \leq \ell \leq \ell_0 + \epsilon$. This gives $\Im(\ell_0+\epsilon) < \mathfrak{w}(\ell_0+\epsilon)$. Furthermore, $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell) \leq \hbar(\ell,\Im(\ell))$ and $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{w}_{+}(\ell) > \hbar(\ell,\mathfrak{w}(\ell))$ for $\ell \in [\ell_0+\epsilon,\ell_0+\tau)$. Next, Theorem 3.3 gives $\Im(\ell) < \mathfrak{w}(\ell)$, $\ell \in [\ell_0,\ell_0+\tau)$.

In a similar way, we can find $\mathfrak{v}(\ell) < \Im(\ell)$, $\ell \in [\ell_0,\ell_0+\tau)$ when $\mathfrak{v}(\ell)$ is an upper function. Consequently, $\mathfrak{v}(\ell) < \Im(\ell) < \mathfrak{w}(\ell)$, $\ell \in [\ell_0,\ell_0+\tau)$. □

4 Existence of solution

In this section, our study investigates the existence and fractal continuation of solutions for an initial value problem related to fractal differential equations that incorporate fractal differential inequalities. To explore the existence of a solution for (1), we will introduce the following definition of equi-fractal continuity.

Definition 4.1 A family of functions $\mathfrak{F} = \{\Im(\ell)\}$ defined on some fractal α-set $\mathcal{E} \subset \mathbb{R}^n$ is said to be equi-fractal continuous if for $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$, independent of $\Im \in \mathfrak{F}$, and also $\ell_1,\ell_2 \in \mathcal{E}$ such that $|\Im(\ell_1) - \Im(\ell_2)| < \epsilon$ whenever $|\ell_1 - \ell_2| < \delta$.

Theorem 4.2 (*Ascoli–Arzela*) *Define* $\mathfrak{F} = \{\Im\}$ *as a sequence of functions established on an* α*-compact set* $\mathcal{E} \subset \mathbb{R}^n$, *which is equi-fractal continuous and equi-fractal bounded. Consequently, there exists a subsequence* $\{\Im_n\}$, $n = 1,2,\ldots$, *that converges* $\mathcal{F}$*-uniformly on* $\mathcal{E}$.

Proof We can prove with similar techniques of the classical Ascoli–Arzela theorem. □

The following theorem represents an existence of solution of (1).

Theorem 4.3 *Let $\hbar \in \mathcal{C}(G, \mathbb{R}^n)$, where G is an α-set*

$$[\ell, \Im):\ \ell_0 \leq \ell \leq \ell_0 + \tau, ||\Im - \Im_0|| \leq b];\ ||\hbar(\ell, \Im)|| \leq M \text{ on } G.$$

Consequently, the initial value problem (1) has at least one solution $\Im(\ell)$ for the interval $\ell_0 \leq \ell \leq \ell_0 + r$, where $r = \min(\tau, \frac{\kappa}{M})$.

Proof Let $\Im_0(\ell)$ be a fractal continuous and fractal differentiable function on $[\ell_0 - \delta, \ell_0]$, $\delta > 0$, such that $\Im_0(\ell_0) = \Im_0$, $||\Im_0(\ell) - \Im)|| \leq \kappa$, and $||\mathcal{D}^{\alpha}_{\mathcal{F}} \Im_0(\ell)|| \leq M$. For $0 < \epsilon \leq \delta$, let $\Im_\epsilon(\ell) = \Im_0(\ell)$ on $[\ell_0 - \delta, \ell_0]$ and

$$\Im_\epsilon(\ell) = \Im_0 + \int_{\ell_0}^{\ell} \hbar(s, \Im_\epsilon(s - \epsilon) d^{\alpha}_{\mathcal{F}} s \tag{4}$$

on $[\ell_0, \ell_0 + r_1]$, where $r_1 = \min(r, \epsilon)$. Clearly, $\Im_\epsilon(\ell)$ is fractal differentiable and

$$||\Im_\epsilon(\ell) - \Im_0|| \leq \kappa \tag{5}$$

on $\ell_0 - \delta$, $\ell_0 + r_1]$. If $r_1 < r$, we can use (4) to extend $u_\epsilon(t)$ as a fractal continuously fractal differentiable function over $[\ell_0 - \delta, \ell_0 + r_2]$, $r_2 = \min(r, 2\epsilon)$ such that (5) holds. Following this approach, $\Im_\epsilon(\ell)$ may be defined across the interval $[\ell_0 - \delta, \ell_0 + r]$ such that it maintains fractal continuity and fractal differentiability, thus satisfying equation (5) within that same interval. Moreover, it follows that $||\mathcal{D}^{\alpha}_{\mathcal{F}} \Im_\epsilon(t)|| < M$, indicating that the sequence $\{\Im_\epsilon(\ell)\}$ constitutes a collection of equi-fractal continuous functions that are uniformly $\mathcal{F}$-bounded. By applying Theorem 4.2, we can observe the existence of a sequence $\{\epsilon_p\}$ such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_p \to 0$ as $p \to \infty$, and the function $\Im(\ell) = \mathcal{F}\text{-}\lim_{p\to\infty} \Im_{\epsilon_p}(\ell)$ exists $\mathcal{F}$-uniformly on $[\ell_0 - \delta, \ell_0 + r]$. Given that $\hbar$ is uniformly $\mathcal{F}$-continuous, it follows that $\hbar(\ell, \Im_{\epsilon_p}(\ell - \epsilon_p))$ converges to $\hbar(\ell, \Im(\ell))$ in an $\mathcal{F}$-uniform manner as p approaches infinity. Consequently, performing term-by-term integration of (4) with $\epsilon = \epsilon_p$ and $r_1 = r$ results in the equation $\Im(\ell) = \Im_0 + \int_{\ell_0}^{\ell} \hbar(s, \Im(s)) d^{\alpha}_{\mathcal{F}} s$. This confirmed that $\Im(\ell)$ serves as a solution for (1). □

Corollary 4.4 *Let $\mathcal{E}$ represent an α-open $(\ell, \Im)$ set within $\mathbb{R}^{n+1}$, and let $\mathcal{E}_0$ denote an α-compact subset of $\mathcal{E}$. Assume that $\hbar$ is an element of $\mathcal{C}(\mathcal{E}, \mathbb{R}^n)$ and that $||\hbar(\ell, \Im)|| \leq M$ throughout $\mathcal{E}$. Equation (1) possesses a solution, and every solution is defined on the interval $[\ell_0, \ell_0 + r]$.*

The subsequent theorem addresses the issue of extending the solutions to the boundary of $\mathcal{E}$. Prior to presenting the theorem, we must first establish the following lemma.

Lemma 4.5 *Let $\hbar$ be a function in $\mathcal{C}(\mathcal{E}, \mathbb{R}^n)$, where $\mathcal{E}$ is an α-open $(\ell, \Im)$ subset of $\mathbb{R}^{n+1}$. Let $\Im(\ell)$ represent a solution to (1) within the interval $\ell_0 \leq \ell < \tau$, $\tau < \infty$. Let us consider a sequence $\{\ell_k\}$ such that $\ell_0 \leq \ell_k$ approaches infinity as k approaches infinity, and the limit $\Im^0 = \mathcal{F}\text{-}\lim_{k\to\infty} \Im(\ell_k)$ exists. If the function $\Im(\ell, \Im)$ remains bounded within the intersection of the set $\mathcal{E}$ and a neighborhood surrounding the point $(\tau, \Im^0)$, then it follows that the limit*

$\mathcal{F}\text{-}\lim_{\ell\to\tau}\Im(\ell)=\Im^0$. *Furthermore, if* $\Im(\tau,\Im^0)$ *is established such that* $\hbar(\ell,\Im)$ *is* $\mathcal{F}$*-continuous at the point* $(\tau,\Im^0)$, *then* $\Im(\ell)$ *is continuously* $\mathcal{F}$*-differentiable with respect to* ℓ *on the interval* $[\ell_0,\tau]$ *and serves as a solution to* (1) *over the interval* $[\ell_0,\tau]$.

Proof Assume that ϵ is a positive quantity that is sufficiently small. Let us take $\widehat{R}:\ 0\le\tau-\ell\le\epsilon, ||\Im-\Im^0||\le\epsilon$ be a given set. Let $M(\epsilon)$ be sufficiently large such that $||\hbar(\ell,\Im)||\le M(\epsilon)$ for $(\ell,\Im)\in\mathcal{E}\cap\widehat{R}$. For sufficiently large values of k, if $0<\tau-\ell_k\le\frac{\epsilon}{2}M(\epsilon)$ and if $||\Im(\ell_k)-\Im^0||\le\frac{\epsilon}{2}$, then

$$||\Im(\ell)-\Im(\ell_k)||<M(\epsilon)(\tau-\ell_k)\le\frac{\epsilon}{2} \tag{6}$$

for $\ell_k\le\ell<\tau$. If this claim does not hold, there exists a time ℓ_1 for which $\ell_k<\ell_1<\tau$, thereby yielding the relationship

$$||\Im(\ell_1)-\Im(\ell_k)||=M(\epsilon)(\tau-\ell_k)\le\frac{\epsilon}{2}.$$

Therefore it follows

$$\begin{aligned}||\Im(\ell)-\Im^0||&\le\frac{\epsilon}{2}||\Im(\ell_k)-\Im^0||\\&\le\epsilon \text{ for } \ell_k\le\ell\le\ell_1.\end{aligned}$$

This indicates that $||\mathcal{D}^\alpha_{\mathcal{F}}\Im(\ell)||\le M(\epsilon)$ for the interval $\ell_k\le\ell\le\ell_1$. Consequently,

$$\begin{aligned}||\Im(\ell_1)-\Im(\ell_k)||&\le M(\epsilon)(\ell_1-\ell_k)\\&<M(\epsilon)(\tau-\ell_k).\end{aligned}$$

This establishes that $\mathcal{F}\text{-}\lim_{\ell\to\tau}\Im(\ell)=\Im^0$, which verifies that (6) is true. The final segment of the lemma is based on the principle that $\mathcal{D}^\alpha_{\mathcal{F}}\Im(\ell)=\hbar(\ell,\Im(\ell))$ converges to $\hbar(\tau,\Im^0)$ when ℓ tends to τ. □

Theorem 4.6 *Let* $\mathcal{E}$ *represent a fractal subset of* $\mathbb{R}^{n+1}$ *that qualifies as an* α*-open* $(\ell,\Im)$*-set in* $\mathbb{R}^{n+1}$. *If* $\hbar$ *is an element of* $\mathcal{C}(\mathcal{E},\mathbb{R}^n)$ *and* $\Im(\ell)$ *is a solution to equation* (1) *over the interval* $\ell_0\le\ell\le\tau_0$, *then it is possible to extend* $\Im(\ell)$ *as a solution to* $\mathcal{E}$*'s boundary.*

Proof Consider the sets $\mathcal{E}_1,\mathcal{E}_2,\ldots$ as α-open subsets of $\mathcal{E}$, where $\mathcal{E}$ can be expressed as the union of $\mathcal{E}_n$; the closures $\overline{\mathcal{E}}_1,\overline{\mathcal{E}}_2,\ldots$ are α-compact, and $\overline{\mathcal{E}}_n$ is contained within $\mathcal{E}_{n+1}$. From Corollary 4.4, it can be concluded that there exists $\epsilon_n>0$ such that if $(\ell_0,\Im_0)$ is contained in $\overline{E}_n$, then all solutions of (1) exist within $\ell_0\le\ell\le\ell_0+\epsilon_n$.

Opt for a large n such that $(\tau_0+\Im(\tau_0))$ is contained within the closure of E_{n_1}. Thereafter, the function $\Im(\ell)$ can be prolonged over the interval $[\tau_0,\tau_0+\epsilon_{n_1}]$, and should the point $(\tau_0+\epsilon_{n_1},\Im(\tau_0+\epsilon_{n_1}))$ belong to the closure $\mathcal{E}_{n_1}$, $\Im(\ell)$ can be additionally extended over the interval $[\tau_0+\epsilon_{n_1},\tau_0+2\epsilon_{n_1}]$. This assertion can be made repeatedly until we secure the extension of $\Im(\ell)$ over the specified interval $\ell_0\le\ell\le\tau_1$ where $\tau_1=\tau_0+N_1\epsilon_{n_1}$, N_1 is an integer greater than or equal to one, such that $(\tau_1,\Im(\tau_1))\notin\overline{\mathcal{E}}_{n_1}$.

Choose n_2 so large that $(\tau_1, \Im(\tau_1)) \in \overline{\mathcal{E}}_{n_2}$. As previously stated, we arrive at an integer $N_2 \geq 1$ such that $u(t)$ may be extended over $t_0 \leq t \leq a_2$, $a_2 = a_1 + N_2\epsilon_{n_2}$ and $(a_2, u(a_2)) \notin \overline{E}_{n_2}$. Following this approach, we arrive at a sequence of integers $n_1 < n_2 < \cdots$ and numbers $\tau_0 < \tau_1 < \cdots$ such that $\Im(\ell)$ has an extension over $[\ell_0, \tau)$, where $\tau = \mathcal{F}\text{-}\lim_{k\to\infty} \tau_k$ and $(\tau_k, \Im(\tau_k)) \notin \overline{\mathcal{E}}_{\tau_k}$. Thus the sequence $\{\tau_k, \Im(\tau_k)\}$ is either unbounded or has a cluster point on the boundary of $\mathcal{E}$. Finally, Lemma 4.5 gives the complete proof. □

Corollary 4.7 *Consider $\hbar \in \mathcal{C}(\mathcal{E}, \mathbb{R}^n)$, wherein $\mathcal{E} = \{(\ell, \Im) : \ \ell_0 \leq \ell \leq \ell_0 + \tau, \ \tau < \infty, \ \Im \text{ defined on } \mathbb{R}^n\}$. Let $\Im(\ell)$ be the solution to (1). Thus, $[\ell_0, \ell_0 + \tau]$ or $[\ell_0, \delta)$, $\delta \leq \ell_0 \leq \ell_0 + \tau$ and $||\Im(\ell)|| \to \infty$ as $\ell \to \infty$ are the biggest intervals of existence of $\Im(\ell)$.*

5 Convergence of the maximal and minimal solution

We shall show minimal and maximal solutions of (1) convergences in the sense of Theorem 4.3 as follows.

Theorem 5.1 *Let $\hbar$ be a function in $\mathcal{C}(R_0, \mathbb{R})$, where R_0 is defined as the fractal-spiral approximated by the rectangle $\ell_0 \leq \ell \leq \ell_0 + \tau$, $|\Im - \Im_0| \leq \kappa$ and $|\hbar(\ell, \Im)| \leq M$ on R_0. Then a comprehensive solution encompassing both maximal and minimal aspects for equation (1) can be determined over the specified interval $[\ell_0, \ell_0 + r]$ with $r = \min\left(\tau, \frac{\kappa}{2M+\kappa}\right)$.*

Proof Let $0 < \epsilon \leq \frac{\kappa}{2}$, and let us consider the fractal differential equation with

$$\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = \hbar(\ell, \Im) + \epsilon, \ \Im(\ell_0) = \Im_0 + \epsilon. \tag{7}$$

It is clear that $\hbar_\epsilon(\ell, \Im) = \hbar(\ell, \Im) + \epsilon$ is properly defined and exhibits $\mathcal{F}$-continuity on R_0, approximated by the interval $\ell_0 \leq \ell \leq \ell_0 + \tau$, $|\Im - (\Im_0 + \epsilon)| \leq \frac{\kappa}{2}$. Clearly, $R_\epsilon \subset R_0$ and $|\hbar_\epsilon(\ell, \Im)| \leq M + \frac{\kappa}{2}$ on R_ϵ. From Theorem 4.3, it is not hard to see that (7) has a solution $\Im(\ell, \epsilon)$ on the interval $[\ell_0, \ell_0 + r]$, where $r = \min\left(\tau, \frac{\kappa}{(2M+\kappa)}\right)$. Next, for $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we have

$$\Im(\ell_0, \epsilon_2) < \Im(\ell_0, \epsilon_1)$$

$$\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell, \epsilon_2) \leq \hbar(\ell, \mathfrak{w}(\ell, \epsilon_2)) + \epsilon_2$$

$$\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell, \epsilon_1) > \hbar(\ell, \Im(\ell, \epsilon_1)) + \epsilon_2, \ \ell \in [\ell_0, \ell_0 + r].$$

Employing Theorem 3.3 leads us to the result that $\Im(\ell, \epsilon_2) < \Im(\ell, \epsilon_1)$, $\ell \in [\ell_0, \ell_0 + r]$. Given that the family of functions $\Im(\ell, \epsilon)$ is equi-$\mathcal{F}$-continuous and uniformly $\mathcal{F}$-bounded on the interval $[\ell_0, \ell_0 + r]$, it can be deduced from Theorem 4.2 that there exists a diminishing sequence $\{\epsilon_n\}$ such that $\epsilon_n \to$ as $n \to \infty$, and the uniform $\mathcal{F}$-limit, specifically $\mathcal{F}\text{-}\lim_{n\to\infty} \Im(\ell, \epsilon_n) = z(\ell)$, is defined on $[\ell_0, \ell_0 + r]$. Clearly, $z(\ell_0) = \Im_0$. The uniform $\mathcal{F}$-continuity of $\hbar$ suggests that $\hbar(\ell, \Im(\ell, \epsilon_n))$ tends uniformly to $\hbar(\ell, z(\ell))$ as $n \to \infty$. Therefore, term-by-term integration applies to $\Im(\ell, \epsilon_n) = \Im_0 + \epsilon_n + \int_{\ell_0}^{\ell} \hbar(s, \Im(s, \epsilon_n) d^{\alpha}_{\mathcal{F}} s$, which further demonstrates that the $\mathcal{F}$-limit $z(\ell)$ is a solution of (1) on $[\ell_0, \ell_0 + r]$. Further, let $\Im(\ell)$ represent any solution of (1) on the interval $[\ell_0, \ell_0 + r]$ then

$$\Im(\ell_0) = \Im_0 < \Im_0 + \epsilon = \Im(\ell_0, \epsilon)$$

$$\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell) < \hbar(\ell, \Im(\ell)) + \epsilon,$$

$$\mathcal{D}^{\alpha}_{\mathcal{F}}\Im(\ell, \epsilon) \geq \hbar(\ell, \Im(\ell, \epsilon)) + \epsilon \text{ for} \ell \in [\ell_0, \ell_0 + r] \text{ and } \epsilon \leq \frac{\kappa}{2}.$$

So, $\Im(\ell) \leq \Im(\ell, \epsilon)$, $\ell \in [\ell_0, \ell_0 + r]$. This shows that the solution is maximal. The uniqueness of the maximal solution indicates that $\Im(\ell, \epsilon)$ converges uniformly to $z(\ell)$ on the interval $[\ell_0, \ell_0 + r]$ as ϵ approaches 0.

In a similar approach, we can find the minimal solution. □

The existence Theorem 5.1, together with the extension Theorem 4.6 implies the following theorem for maximal and minimal solutions of the given initial value problem (3) that may be extended to the boundary of $\mathcal{E}$.

Theorem 5.2 *Let $g \in \mathcal{C}(E, \mathbb{R})$ where E is an α-open (t, u)-set in $\mathbb{R}^2$ and $(t_0, u_0) \in E$. Then* (1) *has maximal and minimal solutions that can be extended to the boundary of E.*

5.1 Comparison theorem

One integral approach in the examination of differential equations that is critical for advancing understanding in this field involves the estimation of a function that complies with a differential inequality by leveraging the external solutions of the related differential equation. In this section of the article, we explore various comparison theorems for fractal differential equations, along with an estimation of a function that meets a fractal differential inequality. The following lemma will play an important role in proving our main theorems.

Lemma 5.3 *Assuming that the hypothesis of Theorem* 5.1 *is valid, let the interval $[\ell_0, \ell_0 + \tau)$ represent the maximum interval for the maximal solution $z(\ell)$ of equation* (3)*. Let us assume that $[\ell_0, \ell_1]$ is an α-compact subinterval of $[\ell_0, \ell_0 + \tau]$. There is then $\epsilon_0 > 0$ such that the maximal solution $z(\ell, \epsilon)$ of equation* (7) *exists over $[\ell_0, \ell_1]$ for $0 < \epsilon < \epsilon_0$, and $\mathcal{F}\text{-}\lim_{\epsilon \to 0} z(\ell, \epsilon) = z(\ell)$ uniformly on $[\ell_0, \ell_1]$.*

Proof For $\ell \in [\ell_0, \ell_1]$, let $\mathcal{E}_0$ be an α-open bounded set, $\mathcal{E}_0 \subset \mathcal{E}$, and $(\ell, z(\ell)) \in \mathcal{E}$. We can choose $\kappa > 0$ such that for $\ell \in [\ell_0, \ell_1]$ the rectangle $R^{\epsilon}_{\ell}: [\ell, \ell + \kappa]$, $|\Im - (z(\ell) + \epsilon)| \leq \kappa$, is included in $\mathcal{E}_0$ for $\epsilon \leq \frac{\kappa}{2}$. Consider $|\hbar(\ell, \Im)| \leq M$ on $\mathcal{E}_0$. Then it is clear that $|\hbar(\ell, \Im) + \epsilon| \leq M + \frac{\kappa}{2}$ on R^{ϵ}_{ℓ} for $\ell \in [\ell_0, \ell_1]$ and $0 < \epsilon \leq \frac{\kappa}{2}$. Let $R^{\epsilon}_{\ell_0}$ be a given rectangle. From Theorem 5.1, the maximal solution $z(\ell, \epsilon)$ of (7) exists on $[\ell_0, \ell_0 + n\nu]$ for $0 < \epsilon < \epsilon_0$, and $\mathcal{F}\text{-}\lim_{\epsilon \to 0} z(\ell, \epsilon) = z(\ell)$ uniformly on $[\ell_0, \ell_0 + 2\nu]$.

By mathematical induction, we can find that there is an $\epsilon_0 = \epsilon_n$ in such a way that $[\ell_0, \ell_1] \subset [\ell_0, \ell_1 + n\nu]$ to ensure that the maximal solution $z(\ell, \epsilon)$ of (7) occurs on $[\ell_0, \ell_0 + n\nu]$ for $0 < \epsilon < \epsilon_0$, and $\mathcal{F}\text{-}\lim_{\epsilon \to 0} z(\ell, \epsilon) = z(\ell)$ uniformly on $[\ell_0, \ell_0 + n\nu]$. □

Theorem 5.4 *Let $\mathcal{E}$ represent an α-open $(\ell, \Im)$-set within $\mathbb{R}^2$ with $\hbar \in \mathcal{C}(\mathcal{E}, \mathbb{R})$. Assume that $[\ell_0, \ell_0 + \tau)$ constitutes the largest interval on which the maximal solution $z(\ell)$ of equation* (3) *is established. Let m be an $\mathcal{F}$-continuous function on the interval $[\ell_0, \ell_0 + \tau)$ mapping to $\mathbb{R}$, where each pair $(\ell, m(\ell))$ belongs to the set $\mathcal{E}$ for $\ell \in [\ell_0, \ell_0 + \tau)$, and the initial value $m(\ell_0)$ satisfies $m(\ell_0) \leq \Im_0$; additionally, this setup is considered with a fixed fractal* *derivative*

$$\mathcal{D}^{\alpha}_{\mathcal{F}} m(\ell) \leq \hbar(\ell, m((\ell)), \ \ell \in [\ell_0, \ell_0 + \tau) - \mathfrak{S}. \tag{8}$$

For all ℓ within the interval $[\ell_0, \ell_0 + \tau)$, it is established that $m(\ell) \leq z(\ell)$.

Proof From Lemma 3.4, it follows that (8) can be replaced by $\mathcal{D}^{\alpha}_{\mathcal{F}} m_{-}(\ell) \leq \hbar(\ell, m(\ell))$, $\ell \in [\ell_0, \ell_0 + \tau)$. Let $\ell_0 < t < \ell_0 + \tau$. By Lemma 5.3, the maximal solution $z(\ell, \epsilon)$ of $\mathcal{D}^{\alpha}_{\mathcal{F}} \Im = \hbar(\ell, \epsilon) + \epsilon$, $\Im(\ell_0) = \Im_0 + \epsilon$ exists on $[\ell_0, t]$ for all $\epsilon > 0$ sufficiently small, and $z(\ell) = \mathcal{F}\text{-}\lim_{\epsilon \to 0} z(\ell, \epsilon)$. Using $\mathcal{D}^{\alpha}_{\mathcal{F}} \Im = \hbar(\ell, \Im) + \epsilon$, $\Im(\ell_0) = \Im_0 + \epsilon$, and $\mathcal{D}^{\alpha}_{\mathcal{F}} m_{-}(\ell) \leq \hbar(\ell, m(\ell))$, $\ell \in [\ell_0, \ell_0 + \tau)$ in Theorem 3.3, we can find $m(\ell) < z(\ell, \epsilon)$, $\ell \in [\ell_0, t]$. □

Remark 5.5 Given the inequality $\mathcal{D}^{\alpha}_{\mathcal{F}} m(\ell) \geq \hbar(\ell, m(\ell))$ alongside the condition $m(\ell) \geq \Im_0$, it follows that $m(\ell)$ must be greater than or equal to $\rho(\ell)$, where $\rho(\ell)$ represents the minimal solution to the differential equation $\mathcal{D}^{\alpha}_{\mathcal{F}} \Im = \hbar(\ell, \Im)$ with the initial condition $\Im(\ell_0) = \Im_0$.

Next, we shall prove Theorem 5.4 under a weaker hypothesis.

Theorem 5.6 *Let $m(\ell)$, $z(\ell)$ be defined as in Theorem 5.4, and consider the set $\mathcal{O} = \{\ell \in [\ell_0, \ell_0 + \tau) : \ z(\ell) < m(\ell) < z(\ell) + \epsilon_0\}$, where $\epsilon_0 > 0$. If the condition $\mathcal{D}^{\alpha}_{\mathcal{F}} m(\ell) \leq \hbar(\ell, m(\ell))$ holds true for ℓ within the subset $\{\ell \in [\ell_0, \ell_0 + \tau) : \ z(\ell) < m(\ell) < z(\ell) + \epsilon_0\} - \mathfrak{S}$, it follows that $m(\ell) \leq z(\ell)$ for ℓ in the interval $[\ell_0, \ell_0 + \tau)$.*

Proof It suffices to demonstrate that $m(\ell) < z(\ell, \epsilon)$ for all ℓ within the interval $[\ell_0, \ell_1]$. According to Lemma 3.4, it follows that $\mathcal{D}^{\alpha}_{\mathcal{F}} m_{-}(\ell) \leq \hbar(\ell, m(\ell))$ for all ℓ within the interval $(\ell_0, \ell_0 + \tau)$. As a result, it follows that $\mathcal{D}^{\alpha}_{\mathcal{F}} m_{-}(\ell) \leq \hbar(\ell, m(\ell))$ for all ℓ within the set $\mathcal{O}$. Next, according to Theorem 3.3, we identify a value ℓ_1 for which the condition $m(\ell_1) = z(\ell_1, \epsilon)$ holds true. Considering the expression $z(\ell) = \mathcal{F}\text{-}\lim_{\epsilon \to 0} z(\ell, \epsilon)$, it is established that there exists a positive value ϵ_0 such that $z(\ell, \epsilon)$ remains less than $z(\ell) + \epsilon_0$ for all ℓ within the interval $[\ell_0, \ell_1]$. Additionally, for all ℓ within the interval $[\ell_0, \ell_1]$, it holds that $z(\ell) < z(\ell, \epsilon) < z(\ell) + \epsilon_0$, and also $z(\ell, \epsilon)) < (z(\ell) + \epsilon_0)$. It follows from $\mathcal{O}$ that $z(\ell_1) < m(\ell_1) < z(\ell_1) + \epsilon_0$. This indicates that ℓ_1 belongs to the set $\mathcal{O}$. Therefore, it holds that $\mathcal{D}^{\alpha}_{\mathcal{F}} m_{-}(\ell) \leq \hbar(\ell, m(\ell))$ for the specified ℓ_1. Consequently, we have $m(\ell) \leq z(\ell)$ for all $\ell \in [\ell_0, \ell_0 + \tau)$. □

Theorem 5.7 *Let $\mathcal{E}$ be an α-open $(\ell, \Im)$-set in $\mathbb{R}^2$, and consider $\hbar \in \mathcal{C}(\mathcal{E}, \mathbb{R})$. Let us consider that $\mathfrak{v}, \mathfrak{w} \in \mathcal{C}([\ell_0, \ell_0 + \tau), \mathbb{R})$, with $(\ell, r(\ell))$ and $(\ell, \mathfrak{w}(\ell))$ belonging to $\mathcal{E}$ where $\ell \in [\ell_0, \ell_0 + \tau)$. Further, suppose $\mathfrak{v}(\ell_0) < \mathfrak{w}(\ell_0)$, and within the interval $(\ell_0, \ell_0 + \tau)$, the inequalities $\mathcal{D}^{\alpha}_{\mathcal{F}} \mathfrak{v}_{-}(\ell) \leq \hbar(\ell, \mathfrak{v}(\ell))$ and $\mathcal{D}^{\alpha}_{\mathcal{F}} \mathfrak{w}_{-}(\ell) \geq \hbar(\ell, \mathfrak{w}(\ell))$ are satisfied for all $\ell \in (\ell_0, \ell_0 + \tau)$. Assume additionally that for each ℓ within the interval $(\ell_0, \ell_0 + \tau)$, and for any ℓ in $[\ell_0, \ell_1]$, the function $\hbar$ meets the specified condition*

$$\hbar(\ell, \Im_1) - \hbar(\ell, \Im_2) \geq -G(\ell_1 + \ell_0 - \ell, \Im_1 - \Im_2), \ \Im_1 \geq \Im_2$$

considering that G is an element of $\mathcal{C}((\ell_0, \ell_0 + \tau) \times \mathbb{R}, \mathbb{R})$ and $z(\ell) \equiv 0$ represents the maximal solution to the equation $\mathcal{D}^{\alpha}_{\mathcal{F}} \Im = G(\ell, \Im)$ with the initial condition $\Im(\ell_0) = 0$, it follows that $\mathfrak{v}(\ell) < \mathfrak{w}(\ell)$ for ℓ in the interval $[\ell_0, \ell_0 + \tau)$.

Proof Similar to the demonstration of Theorem 3.3, there exists ℓ_1 within the interval $(\ell_0, \ell_0 + \tau)$ for which the condition $\mathfrak{v}(\ell_1) = \mathfrak{w}(\ell_1)$ holds true, while $\mathfrak{v}(\ell) < \mathfrak{w}(\ell)$ for all ℓ satisfying $\ell_0 \leq \ell < \ell_1$. Define the functions $\mathfrak{v}_1(\ell) = \mathfrak{v}(\ell_1 + \ell_0 - \ell)$ and $\mathfrak{w}_1(\ell) = \mathfrak{w}(\ell_1 + \ell_0 - \ell))$. Given that $\mathfrak{v}(\ell_1) = \mathfrak{w}(\ell_1)$ and $\mathfrak{v}(\ell) < \mathfrak{w}(\ell)$ for $\ell_0 \leq \ell < \ell_1$, it follows that $\mathfrak{v}_1(\ell_0) = \mathfrak{w}_1(\ell_0)$ and $\mathfrak{v}_1(\ell) \leq \mathfrak{w}_1(\ell)$ for $\ell \in [\ell_0, \ell_1]$.

Let $m(\ell) = \mathfrak{w}_1(\ell) - \mathfrak{v}_1(\ell)$, the definitions of $\mathfrak{v}_1$, $\mathfrak{w}_1$ and assumptions $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{-}(\ell) \leq \hbar(\ell, \mathfrak{v}(\ell))$ and $\mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{w}_{-}(\ell) \geq \hbar(\ell, \mathfrak{w}(\ell))$ for $\ell \in (\ell_0, \ell_0, \tau)$ imply the inequality

$$\begin{aligned}\mathcal{D}^{\alpha}_{\mathcal{F}}m_{-}(\ell) &= \mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{w}_{1-}(\ell) - \mathcal{D}^{\alpha}_{\mathcal{F}}\mathfrak{v}_{1-}(\ell)\\ &\leq \hbar_1(\ell, \mathfrak{w}_1(\ell)) - \hbar_1(\ell, \mathfrak{v}_1(\ell))\end{aligned}$$

wherever $\hbar_1(\ell, \Im) = -\hbar(\ell_1 + \ell_0 - \ell, \Im)$. Since $\mathfrak{v}_1(\ell) \leq \mathfrak{w}_1(\ell)$, $\ell \in [\ell_0, \ell_1]$ holds, we can use $\hbar(\ell, \Im_1) - \hbar\big((\ell, \Im_2) \geq -G(\ell_0 + \ell_1 - \ell, \Im_1 - \Im_2\big)$, $\Im_1 \geq \Im_2$ to arrive at $\mathcal{D}^{\alpha}_{\mathcal{F}}m_{-}(\ell) \leq G\big(\ell, m(\ell)\big)$, $\ell \in [\ell_0, \ell_1]$. According to Theorem 5.4, it is established that $m(\ell) \leq z(\ell)$ for all ℓ within the interval $[\ell_0, \ell_1]$, where $z(\ell)$ represents the maximal solution to the differential equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = G(\ell, \Im)$ with the initial condition $z(\ell_0) = m(\ell_0)$. Based on the given definition of $m(\ell)$, we observe that $\mathfrak{v}_1(\ell_0)$ equals $\mathfrak{w}_1(\ell_0)$ and $\mathfrak{v}_1(\ell) \leq \mathfrak{w}_1(\ell)$ for ℓ in the interval $[\ell_0, \ell_1]$. Consequently, it follows that $m(\ell) \geq 0$ within the same interval, with the specific condition that $m(\ell_0) = 0$. The inequality $m(\ell) \leq z(\ell)$ for $\ell \in [\ell_0, \ell_1]$, along with the assumption that $z(\ell) \equiv 0$, indicates that $\mathfrak{v}(\ell) = \mathfrak{w}(\ell)$ for $\ell \in [\ell_0, \ell_1]$. However, this directly contradicts the assumption that $\mathfrak{v}(\ell) < \mathfrak{w}(\ell)$ for $\ell_0 \leq \ell < \ell_1$. Consequently, the set $\mathcal{O}$ is empty, thereby proving the theorem. □

A more comparison theorem of Theorem 5.4 is stated below.

Theorem 5.8 *Consider the product space $\mathcal{E} = [\ell_0, \ell_0 + \tau) \times \mathbb{R}^2$ with a function $\hbar \in \mathcal{C}(\mathcal{E}, \mathbb{R})$. Assume that $\hbar$ is nondecreasing with respect to ν for every fixed ℓ and $\Im$. Suppose that $z(\ell)$ represents the maximal solution of the fractal differential equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = \hbar(\ell, \Im)$ subject to the initial condition $\Im(\ell_0) = \Im_0 \geq 0$, which is valid on the interval $[\ell_0, \ell_0 + \tau)$, and that $z(\ell) = 0$ for ℓ within this interval. Consequently, the maximal solution $z_1(\ell)$ for the equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = g_1(\ell, \Im)$, with $\Im(\ell_0) = \Im_0 > 0$ and where $\hbar_1(\ell, \Im) = \hbar(\ell, z(\ell))$, also exists on $[\ell_0, \ell_0 + \tau)$, ensuring $z(\ell) = z_1(\ell)$ for ℓ within the interval considered.*

Proof According to Theorem 4.3 and Theorem 4.6, the maximal solution $z_1(\ell)$ of the differential equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = g_1(\ell, \Im)$, with the initial condition $\Im(\ell_0) = \Im_0 > 0$, exists within the interval $[\ell_0, \ell_0 + r]$, where $r < \tau$, and this solution can be extended to reach the boundary of $\mathcal{E}$. This indicates that the function $z_1(\ell)$ is valid within the interval $[\ell_0, \ell_0 + \tau)$, or alternatively, there exists a point $\ell_1 < \ell_0 + \tau)$ at which the magnitude of $z_1(\ell_k)$ approaches infinity for a specific sequence ℓ_k as ℓ_k converges to l when k tends toward infinity. Observe that $\mathcal{D}^{\alpha}_{\mathcal{F}}z(\ell) = \hbar(\ell, z(\ell)) = \hbar_1(\ell, z(\ell))$, and this yields, from Theorem 5.4, that $z(\ell) \leq z_1(\ell)$ as far as $z_1(\ell)$ exists. It follows from $z(\ell) = 0$, $\ell \in [\ell_0, \ell_0 + \tau)$, $|z_1(\ell_k)| \to \infty$, and $z(\ell) \leq z_1(\ell)$ that $z_1(\ell_k) \to +\infty$ as $\ell_k \to l$. Consider the maximal solution $z(\ell, \epsilon)$ to the differential equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = \hbar(\ell, \Im) + \epsilon$ with the initial condition $\Im(\ell_0) = \Im_0 + \epsilon$, where $\Im_0 \geq 0$. According to Lemma 5.3, this solution is defined on the interval $[\ell_0, \ell_1 + \nu]$, with $\nu > 0$ and $\ell_1 + \nu < \ell_0 + \tau$, for sufficiently small $\epsilon > 0$. Furthermore, given the equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = \hbar(\ell, \Im) + \epsilon$ and the inequality $z(\ell_0) < z(\ell_0, \epsilon)$, it follows that $\mathcal{D}^{\alpha}_{\mathcal{F}}z(\ell, \epsilon) > \hbar(\ell, z(\ell, \epsilon)$ and $z(\ell_0) < z(\ell_0, \epsilon)$. Consider the maximal solution $z(\ell,\epsilon)$ to the differential equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = \hbar(\ell,\Im) + \epsilon$ with the initial condition $\Im(\ell_0) = \Im_0 + \epsilon$, where $\Im_0 \geq 0$. According to Lemma 5.3, this solution is defined on the interval $[\ell_0, \ell_1 + \nu]$, with $\nu > 0$ and $\ell_1 + \nu < \ell_0 + \tau$), for sufficiently small $\epsilon > 0$. Furthermore, given the equation $\mathcal{D}^{\alpha}_{\mathcal{F}}\Im = \hbar(\ell,\Im) + \epsilon$ and the inequality $z(\ell_0) < z(\ell_0,\epsilon)$, it follows that $\mathcal{D}^{\alpha}_{\mathcal{F}}z(\ell,\epsilon) > \hbar(\ell, z(\ell,\epsilon))$ and $z(\ell_0) < z(\ell_0,\epsilon)$. Additionally, it is established that $\mathcal{D}^{\alpha}_{\mathcal{F}}z(\ell) = \hbar_1(\ell, z_1(\ell))$. Considering the monotonic nature of $\hbar$ in relation to $\mathfrak{v}$ and the condition $z(\ell) \leq z_1(\ell)$, it follows that $\mathcal{D}^{\alpha}_{\mathcal{F}}z_1(\ell) \leq \hbar(\ell, z_1(\ell))$. The application of Theorem 5.4 yields $z_1(\ell) \leq z(\ell)$ for $\ell \in [\ell_0, \ell_0 + \epsilon)$. Consequently, the combination of this inequality with $z(\ell) \leq z_1(\ell)$ confirms the equality $z(\ell) = z_1(\ell)$ for $\ell \in [\ell_0, \ell_0 + \tau)$. □

Remark 5.9 Fractal differential inequalities, merging fractal calculus with inequalities, have surfaced as significant tools for representing complex systems in multiple areas. They are especially advantageous in cases where standard integer-order models are insufficient, particularly in systems exhibiting memory, nonlocality, or fractal geometry. These inequalities assist in analyzing the behavior of solutions to fractal differential equations, yielding important insights regarding stability, existence, and bounds. In our next work, as a real world applications, we shall investigate modelling complex systems. These equations, which integrate fractal calculus, are employed to simulate phenomena such as water permeation in wool fibers, heat transfer in multi-scale fabrics, and various other intricate systems defined by nonlocality and memory effects.

6 Conclusion

This paper investigates the realm of fractal differential inequalities, presenting a rigorous exploration on the existence of solutions for fractal differential equations with initial values. Employing a methodology akin to Picard's approximation, we introduce a refined approach using rectangular approximations to analyze fractal spirals. Our study addresses both maximal and minimal solutions pertaining to the proposed initial value fractal differential equations. Additionally, we delineate a suite of comparison theorems pertinent to fractal differential equations and offer an estimation for functions that adhere to fractal differential inequalities.

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Data availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mechanical Engineering, College of Engineering, Qassim University, Buraydah, Saudi Arabia. [2]Mathematics Division, School of Advanced Sciences and Languages, VIT Bhopal University, Bhopal-Indore Highway, India. [3]Department of Mathematics, Usak University, Usak, Turkey. [4]Department of Mathematics, Mbeya University of Science and Technology, Mbeya, Tanzania.

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