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Fixed point theorems for interpolative orthogonal relational in TVS-valued cone metric spaces

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Abstract

This article explores the fixed point theorem for a novel class of interpolative relation theoretical convex mappings in TVS-valued cone metric spaces, integrating relational theory, convexity, and interpolation properties to offer fresh perspectives and possible uses in theoretical and applied mathematics. An application of the results to differential equations and matrix equations in the context of orthogonal TVS-valued cone metric spaces is presented, along with a constructive example to support the findings.

Mathematics Subject Classification: 47H10; 54H25

Keywords: Fixed point theorems; Interpolative mapping; Orthogonal Relation; TVS-valued Cone Metric Spaces; Differential equation; Matrix equation

1 Introduction

An element that the function maps to itself is called a fixed point. Fixed-point theory a key tool in nonlinear analysis and many other areas of contemporary mathematics. Specifically, we define the problem in terms of identifying a fixed point of a given mapping [4] when we address the solvability of a certain functional equation (differential equation, fractional differential equation, integral equation, and matrix equation). Applications and ramifications of fixed point theory can be found in many disciplines, including biology, chemistry, economics, dynamical systems, physics, optimization, game theory, and chemistry.

Theorem 1 *Let (X,d) be a complete metric space, and let Υ be a contraction in X, i.e., there exists $\sigma \in [0,1)$ such that*

$$d(\Upsilon x, \Upsilon y) \leq \sigma d(x,y) \tag{1}$$

for all $x,y \in X$. Then, Υ has a unique fixed point.

The theorem has become a fundamental source of information for researchers studying nonlinear analysis and its applications in various mathematical and scientific fields.

Several generalizations of the concept of a metric space exist in the literature. Huang and Zhang [14] introduced the concept of normal cone metric spaces and established fixed point theorems for mappings satisfying Banach contraction and Kannan-type contractive conditions. However, Rezapour and Hamlbarani [33] demonstrated that normality is not a required condition and provided examples of non-normal cones. By refining the normality criteria proposed by Huang and Zhang [14]. Beg *et al.* [5] explored common fixed points for a pair of self-mappings under a generalized contractive condition. Their work did not assume normality and was conducted within a broader class of topological vector space-valued (TVS) cone metric spaces, extending the framework introduced by Huang and Zhang [14].

Istrățescu [15] introduced the concept of convex contraction and demonstrated that every convex contraction on a complete metric space possesses a unique fixed point. This result extends the Banach contraction principle. Alghamdi *et al.* [2] used this concept to obtain a generalization of the Banach contraction principle to the class of convex contractions on non-normal cone metric spaces by extending the recent results of Huang and Zhang [14] and Rezapour and Hamlbarani [33].

In 2017, Gordji *et al.* [13] established results concerning orthogonal sets and extended the Banach contraction principle. Furthermore, they presented applications for their results to ensure the existence and uniqueness of solutions for first-order differential equations. Gnanaprakasam *et al.* [12] introduced new fixed point results in orthogonal b-metric spaces with associated applications. Mani *et al.* [23] proved the results on orthogonal coupled fixed point results with an application in orthogonal metric space. Javid *et al.* [16] introduced the concept of an orthogonal partial b-metric and established some fixed point theorems for related contractions with an application for the Volterra integral equation. Javid *etal.* [17] investigate the conditions for the existence of fixed-point for generalized contractions in the orthogonal extended b-metric spaces endowed with an arbitrary binary relation with application to the system of boundary-valued problem. Uddin *et al.* [36] they initiated the concept of orthogonal m-metric spaces and an application to solve Fredholm integral equations. Mani *et al.* [24] proved common fixed point theorems in orthogonal Branciari metric spaces with an application. Bilgili Gungor [8] extended the concept of orthogonal p-contraction in orthogonal metric spaces and related nonlinear problems using w-distance and also find a solution to nonlinear Fredholm integral equations. Prakasam *et al.* [31] investigated an orthogonal L^*-contraction map concept and proved the fixed point theorem in an orthogonal complete Branciari metric space and demonstrated the existence of a uniqueness solution to the fourth-order differential equation. Nallaselli *et al.* [26] proved fixed point theorems via orthogonal convex contraction in orthogonal b-metric spaces with applications.

Interpolation theory involves estimating or constructing intermediate values or spaces based on known data or structures. In the context of functional analysis, interpolation is about estimating norms or constructing spaces that lie "between" two given Banach spaces or other functional spaces. These inequalities are widely used in analysis, particularly in partial differential equations, Sobolev spaces, functional analysis, partial differential equations, and vibrating string problems, termed boundary-valued problems.

Gagliardo [11] and Nirenberg [29] independently established the following interpolation inequality:

$$\|\nabla_u^l\|_{L_p(\mathbb{R}^d)} \leq \|\nabla_u^k\|^{\frac{1}{k}}_{L_p(\mathbb{R}^d)} \|u\|^{1-\frac{1}{k}}_{L_r(\mathbb{R}^d)}, \tag{2}$$

with $0 \leq l < k, \frac{k}{p} = \frac{l}{r} + \frac{k-l}{q}$ for $1 \leq r < \infty$, $1 \leq q < \infty$ and $u \in W^{k,r}(\mathbb{R}^d)$.

This inequality is called the Gagliardo-Nirenberg inequality.

The Gagliardo-Nirenberg inequality interpolates between L^p norms of a function and its derivatives. For example, in one dimension

$$\|u\|_{L^p} \leq C\|u'\|^{\theta}_{L^q}\|u\|^{1-\theta}_{L^r},$$

where $1 \leq r, q, p \leq \infty$, $\frac{1}{p} = \theta\left(\frac{1}{q} - 1\right)\frac{1-\theta}{r}$, $\theta \in [0, 1]$.

The inequality is used to controll nonlinear terms in PDEs.

Suppose we analyze the nonlinear term u^3 in the context of the heat equation as follows:

$$u_t - \Delta u = u^3, \quad u(x,t) \in \mathbb{R}.$$

To estimate the nonlinear term u^3, we need a bound on $\|u^3\|_{L^2}$. Using the Gagliardo-Nirenberg inequality, we can interpolate between the L^6 norm of u (arising from Sobolev embedding theorems) and the L^2 norm of u.

Using $p = 6$, $q = 2$, and $r = 2$ in the inequality, we get

$$\|u\|_{L^6} \leq C\|\nabla u\|^{\theta}_{L^2}\|u\|^{1-\theta}_{L^2},$$

where $\theta = \frac{1}{3}$.

Since $\|u^3\|_{L^2} = \|u\|^3_{L^6}$, we substitute the interpolation result as follows:

$$\|u^3\|_{L^2} \leq C\left(\|\nabla u\|^{\frac{1}{3}}_{L^2}\|u\|^{\frac{2}{3}}_{L^2}\right)^3 = C\|\nabla u\|_{L^2}\|u\|^2_{L^2}.$$

This bound is crucial in establishing well-posedness or energy estimates for the PDE, as it provides a way to control the growth of the nonlinear term u^3 in terms of the L^2 norm of u and its gradient ∇u.

In the framework of interpolation, we consider a pair of Banach spaces (X_0, X_1) (or more generally, quasi-Banach spaces) and define intermediate spaces $(X_0, X_1)_{\theta,q}$, where $0 < \theta < 1$ and $1 \leq q \leq \infty$. These intermediate spaces "interpolate" between X_0 and X_1 in a precise sense that depends on the parameters θ and q. Bergh and Löfström [6] and Krein *et al.* [22] proved the following interpolation inequality:

$$\|Tx\|_{\theta,q} \leq \|T_0\|^{1-\theta}_{\theta,q}\|T_1\|^{\theta}_{\theta,q}, \tag{3}$$

where T is a bounded linear operator, and T_0 and T_1 denote the restrictions of T to X_0 and X_1, respectively. This inequality holds under the assumption that T is bounded as an operator from $X_0 \to Y_0$ and from $X_1 \to Y_1$, where $(X_0, X_1)_{\theta,q}$ and $(Y_0, Y_1)_{\theta,q}$ are the interpolation spaces for the input and output pairs, respectively. This inequality contains the idea

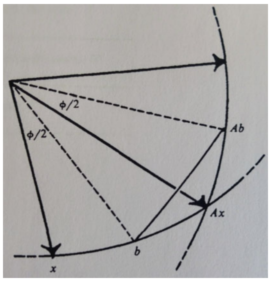

Figure 1 An orthogonal plane

that the norm of T on the interpolated space $(X_0,X_1)_{\theta,q}$ can be controlled by the norms of T on X_0 and X_1. This is particularly useful when working with spaces like L^p spaces, Sobolev spaces, or general Banach spaces, where interpolation theory provides a way to extend results from two "endpoint" spaces to the entire family of intermediate spaces.

Additionally, Karapinar [19] modified the classical Kannan [18] mapping using the interpolative concept. The result for an interpolative Reich-Rus-Ćirić-type contraction on partial metric spaces was provided by Karapinar *et al.* [20]. The fixed point, its geometry, and its application by ω-interpolative contraction of Suzuki-type mapping were demonstrated by Tomar *et al.* [35]. Nazam *et al.* [27] introduce the (Ψ,ψ)-orthogonal interpolative contraction as a generalization of an orthogonal interpolative contraction with an application to solve a fractional differential equation. Dhanraj *et al.* [10] established a fixed point theorem on orthogonal extended interpolative ψ-$\mathcal{F}$-contraction.

Buser [9] showed a geometric approach to invariant subspace of orthogonal matrices.

An $n \times n$ orthogonal matrix A with real entries can be transformed into a normal form consisting of 2×2 and 1×1 blocks. This implies that $\mathbb{R}^n$ can be decomposed into an orthogonal sum of 1- or 2-dimensional subspaces that are invariant under A. Typically, this is shown through induction and polynomial factorization. However, a purely geometrical argument can also identify a subspace for the induction process, demonstrating the invariance under A. If the orthogonal matrix A has real eigenvalues, we are done. If not, there exists some such that $x \in \mathbb{R}^n$, x and Ax are not parallel; these vectors determine a plane. Since the angle between x and Ax can be considered a distance on the compact surface of the unit sphere, there is an $x \in \mathbb{R}^n$ for which the angle θ between x and Ax is minimized. Let b be the bisector of the angle between x and Ax (any other vector in the same plane would also suffice). As A is orthogonal see Fig. 1, it preserves angles; hence, the triangle inequality implies

$$\phi \leq (b,Ab) \leq (b,Ax) + (Ax,Ab) = \frac{\phi}{2} + \frac{\phi}{2}. \tag{4}$$

Clearly, the inequalities must be equalities, which forces Ab to lie in the same plane as b, x, and Ax. Hence, x and b span a two-dimensional invariant subspace.

According to the literature mentioned above, numerous studies have explored related ideas, such as interpolation theory, orthogonality in metric spaces, and cone metrics. While these publications provide crucial background information, none have specifically examined the interaction of interpolative orthogonality in TVS-valued cone metric spaces. To address this gap, we present a comprehensive analysis of these theoretical properties and practical applications of these components, along with a novel approach to integrating them in this work.

This study explores fixed point theorems for interpolative orthogonal relational convex mappings in TVS-valued cone metric spaces, an extension of traditional metric spaces incorporating vector structures and orthogonal cone properties. Inspired by works of notable researchers such as Karapinar [19], Gordji *et al.* [13], Nazam *et al.* [28] Alghamdi *et al.* [2], Rezapour and Hamlbarani [33], Istrățescu [15], Miandaragh *et al.* [25], and Alam and Imdad [1], the paper generalizes existing results and develops new fixed point theorems, focusing on applications in differential and matrix equations. The research bridges gaps in the literature by addressing the under-explored interaction between orthogonality and interpolation in these advanced mathematical structures, offering a novel framework with implications for optimization theory, nonlinear analysis, and functional analysis. Additionally, we add a nontrivial example to this study that uses an orthogonal interpolative convex cone TVS-valued metric, which is not a TVS-valued metric. We established a few theories regarding the suggested construction.

2 Preliminaries

We begin with key definitions commonly used in fixed point theory, which are essential for a clear understanding of the subsequent results.

An orthogonal TVS-valued cone metric space is a generalization of a metric space where the distance function takes values in a topological vector space (TVS) instead of the real numbers, and the concept of orthogonality is introduced.

Definition 1 [5] Let (E,τ) be always a topological space, and let P be a subset of E. Then, P is called a cone whenever:

(i) P is closed, non-empty, and $P \neq \{0\}$;
(ii) $ax + by \in P$ for all $x, y \in P$ and nonnegative real number a, b;
(iii) $P \cap (-P) = \{0\}$.

Cone $P \subseteq E$ is presented. We say that P is solid if its interior (int P) is non-empty. The notation $x \ll y$ is used to indicate that $y - x \in intP$ if P is a solid cone, and P is a component of P. Keep in mind that for every $x, y, z \in intP$, $x \ll z$ if $x \ll y$ and $y \leq z$.

Using the concepts introduced by Gordji *et al.* [13], Rezapour and Hamlbarani [33], and Beg *et al.* [5], we establish some properties that satisfy an orthogonal TVS-valued cone metric space.

Definition 2 Let $(X, \perp)$ be a non-empty O-set. Suppose that the mapping $d : X \times X \to E$, satisfies:

(i) $0 \preceq d(x,y)$, for all $x \perp y \in X$ and $d(x,y) = 0_E$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$, for all $x \perp y \in X$;
(iii) $d(x,y) \preceq d(x,z) + d(z,y)$, for all $x, y, z \in X$such that $x \perp y, y \perp z, x \perp z \in X$,

where 0_E is the zero element in E, and $\preceq$ is a partial order on E. Additionally, the orthogonality condition is introduced, where a relation $\perp$ on X satisfies specific properties. Then, d is called a cone metric on X, and $(X, \perp, d)$ is called orthogonal topological vector valued cone metric space.

Example 1 Let $E = (C_{[0,1]}, \mathbb{R}^2), P = \{(x,y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $x \perp y$ and $d(x,y) = |x - y|\psi(t)$, where $\psi(t) = e^t$. Then, (X, E) is an orthogonal TVS-valued cone metric space.

Below, we define the concepts of completeness, orthogonal continuity, and TVS-valued cone orthogonal sequence.

Definition 3

(i) Consider the orthogonal set $(X,\perp)$ (O-set). Then, a sequence $\{x_n\}$ is considered orthogonal (or an O-sequence) if

$$(\forall\, n \in \mathbb{N}) x_n \perp x_{n+1} \vee x_{n+1} \perp x_n.$$

(ii) A sequence $\{x_n\}$ is referred to as a Cauchy sequence and an O-sequence if

$$(\forall\, n \in \mathbb{N}) x_n \perp x_{n+1} \vee x_{n+1} \perp x_n.$$

Definition 4 Consider the binary relation $(X,\perp)$ defined on $X \times X$. If there exists $x_0 \in X$ such that $(X,\perp)$ is an orthogonal set (O-set), then

$$\forall\, y \in X, x_0 \perp y \vee \forall\, y \in X, y \perp x_0.$$

The element x_0 is called an orthogonal element. An orthogonal set may have more than one orthogonal element.

Definition 5

(i) An orthogonal set (O-set) is denoted by $(X,\perp)$. An orthogonal relationship between any two elements $x, y \in X$ is defined as $x \perp y$.

(ii) Let $(X,\perp)$ be an orthogonal set (O-set), and let d be a cone metric on X with a TVS value. In this case, $(X,\perp,d)$ is referred to as an orthogonal TVS-valued cone metric space.

Definition 6 If every Cauchy O-sequence converges in X, then the orthogonal TVS-valued cone metric space $(X,\perp,d)$ is a complete 0-TVS-valued cone metric space.

Definition 7 Let $(X,\perp,d)$ be an orthogonal cone metric space with TVS values. If for O-sequence $\{x_n\}$ converges to x, then $\Upsilon x_n \to \Upsilon x$ as $n \to \infty$, then a function $\Upsilon : X \to X$ is orthogonally continuous (O-continuous) at x.

Definition 8 Orthogonality($\perp$-preserving) is the state of a self-mapping Υ on O-TVS-valued cone metric space if $x \perp y$ implies $\Upsilon x \perp \Upsilon y$ for all $x, y \in X$.

Example 2 [13] Assume that $X = [0,1]$ and the Euclidean metric is the metric on X. For every $x, y \in X$, define $x \perp y$ if $x, y \in \{x, y\}$. Assume that the mapping $\Upsilon : X \to X$ is defined by

$$\Upsilon x = \begin{cases} \frac{x}{2}, & x \in \mathbb{Q} \cap X, \\ 0, & x \in \mathbb{Q}^c \cap X. \end{cases} \tag{5}$$

Then, Υ is a $\perp$-preserving mappings.

The following are preliminary results.

Krein *et al.* [22] introduced the concept of interpolation in Banach spaces. For the interpolative Kannan contraction, Karapinar [19] presented the following results:

Definition 9 [19] Let (X,d) be a metric space. The mapping $\Upsilon : X \to X$ is referred to as an interpolative mapping of the Kannan contractions if

$$d(\Upsilon x, \Upsilon y) \leq c[d(x, \Upsilon x)]^{\delta}.[d(y, \Upsilon y)]^{1-\delta}, \tag{6}$$

for all $x,y \in X$ with $x \neq \Upsilon x$, where $c \in [0,1)$ and $\delta \in (0,1)$.

Theorem 2 [19] *Suppose that* (X,d) *is a complete metric space and* Υ *is an interpolative contraction of the Kannan type. Then, in* X, Υ *has a unique fixed point.*

The following definition was recently provided by Karapinar *et al.* [21].

Definition 10 [21] Let $\Upsilon : X \to X$, and let $\alpha : X \times X \to (-\infty,\infty)$. We say that Υ is a triangular α-admissible mapping if

(T1) $\alpha(x,y) \geq 1 \implies \alpha(\Upsilon x, \Upsilon y) \geq 1, x,y \in X,$

(T2) $\begin{cases} \alpha(x,z) \geq 1 \\ \alpha(z,y) \geq 1, \implies \alpha(x,y) \geq 1. \end{cases}$

In convex metric spaces, Istrățescu [15] established the following definition.

Definition 11 A continuous mapping $\Upsilon : X \to X$ is called a convex contraction of order 2 if there exist constants $a,b \in (0,1)$ such that for all $x,y \in X$, the following holds:

$$d(\Upsilon^2 x, \Upsilon^2 y) \leq ad(\Upsilon x, \Upsilon y) + bd(x,y),$$

and $a + b < 1$.

Istrățescu [15] also presented the following example to illustrate the proven results.

Example 3 Let $X = [0,1]$ with the usual metric $\Upsilon : X \to X$ by the relation

$$\Upsilon = \frac{1}{2}\left(x^2 + \frac{1}{2}\right),$$

for all $x,y \in X$.

Alghamdi *et al.* [2] supplied the following definitions and theorems in cone convex metric space.

Definition 12 [2] A continuous mapping $\Upsilon : X \to X$ defined on a cone metric space X is called a cone convex contraction mapping of order 2 if there exist constants $a,b \in (0,1]$ such that

$$d(\Upsilon^2 x, \Upsilon^2 y) \preceq ad(\Upsilon x, \Upsilon y) + bd(x,y),$$

for all $x,y \in X$, and $a + b < 1$.

Theorem 3 [2] *Let* (X, d) *be a complete cone metric space over a solid cone* P*. Let* $\Upsilon : X \to X$ *be a cone convex contraction mapping of order* 2*. Then,* Υ *has a unique fixed point in* X*, and for any* $x \in X$*, the iterative sequence* $\{\Upsilon^n(x)\}$ *converges to this fixed point.*

Definition 13 [2] A two-sided cone convex contraction mapping of order 2 is a continuous mapping $\Upsilon : X \to X$ defined on a cone metric space X if there are $a_1, a_2, b_1, b_2 \in (0, 1]$ such that

$$d(\Upsilon^2 x, \Upsilon^2 y) \preceq a_1 d(x, \Upsilon x) + a_2 d(\Upsilon x, \Upsilon^2 x) + b_1 d(y, \Upsilon y) + b_2 d(\Upsilon y, \Upsilon^2 y),$$

for all $x, y \in X$, and $a_1 + a_2 + b_1 + b_2 < 1$.

Theorem 4 [2] *Let* (X, d) *be a complete cone metric space over a solid cone* P*, and let* $\Upsilon : X \to X$ *be a cone convex contraction mapping of order* 2*. Then,* Υ *has a unique fixed point in* X*. Moreover, for any* $x \in X$*, the iterative sequence* $\{\Upsilon^n x\}$ *converges to this fixed point.*

The following lemma was introduced by Berinde [7] for an asymptotically regular map to have the approximate fixed point property.

Lemma 5 [7] *Assume that* (X, d) *is a metric space and* Υ *is an asymptotically regular self-mapping on* X*, that is*

$$d(\Upsilon^i x, \Upsilon^{i+1} x) \to 0,$$

for all $x \in X$*. Then,* Υ *has the approximate fixed point property.*

In an orthogonal metric space, Gordji *et al.* [13] established the following definition and theorem, extending the Banach contraction principle [4] and the results obtained by Ran and Reurings [32].

Definition 14 Suppose that $0 < \lambda < 1$ and $(X, \perp, d)$ is an orthogonal metric space. A mapping $\Upsilon : X \to X$ is considered an orthogonal contraction with a Lipschitz constant λ, if, for any $x, y \in X$ with $x \perp y$, the following inequality holds

$$d(\Upsilon x, \Upsilon y) \leq \lambda d(x, y).$$

Theorem 6 *Let* $0 < \lambda < 1$ *and* $(X, \perp, d)$ *be an orthogonal complete metric space. Let* $\Upsilon : X \to X$ *be an orthogonal contraction that preserves* $\perp$ *and the Lipschitz constant* λ*. There is a single fixed point* $x^\star \in X$ *for* Υ*. Additionally,* Υ *is a Picard operator, meaning that for any* $x \in X$*,* $\lim_{n\to\infty} \Upsilon^n x = x^\star$*.*

Examples that meet the above theorem [13] include the following.

Example 4

(i) Assume that Q is a positive definite matrix and $\mathcal{M}(n)$ is the set of all $n \times n$ matrices. The relationship $\perp$ on $\mathcal{M}(n)$ is defined by

$$A \perp B \Longleftrightarrow \exists X \in \mathcal{M}(n) : AX = B.$$

It is easy to see that $I \perp B, B \perp 0$ and $Q^{\frac{1}{2}} \perp B$.

(ii) Examine the orthogonal relation $\perp_C$ on $\mathcal{M}(n)$ with respect to C for $C \in \mathcal{M}(n)$ given by

$$A \perp_C B \Longleftrightarrow tr(ABC) = tr(CBA).$$

Keep in mind that for every $B \in \mathcal{M}(n)$, $C \perp_C B$.

(iii) In a Euclidean space, let X be the inner product $\mathbb{R}^n$. To get the inner product of two vectors, $\mathbf{x} = [x_1, x_2, \dots, x_n]$ and $\mathbf{y} = [y_1, y_2, \dots, y_n]$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i = 0.$$

It is easy to see that $0 \perp_x$ for all $x \in X$, and hence $(X, \perp)$ is a 0-set.

(iv) Let $X = \mathbb{R}$ and suppose that $x \perp y$ if

$$x, y \in \left(n + \frac{1}{3}, n + \frac{2}{3}\right)$$

for some $n \in z$. $(X, \perp)$ is an O-set.

Berinde [7] introduced an almost contraction maps as follows.

Definition 15 Let (X, d) be a space with metrics. When a constant $\sigma \in (0,1)$ and certain $\lambda \geq 0$ exist, a mapping $\Upsilon : X \to X$ is referred to as an almost contraction.

$$d(\Upsilon x, \Upsilon y) \leq \sigma d(x,y) + \lambda d(y, \Upsilon x). \tag{7}$$

The following example satisfies the almost contraction condition.

Example 5 Consider the mapping $\Upsilon : \mathbb{R} \to \mathbb{R}$ defined on real numbers $\mathbb{R}$, where $\mathbb{R}$ is equipped with the usual metric $d(x,y) = |x - y|$:

$$\Upsilon x = \frac{x}{2}.$$

Gnanaprakasam *et al.* [12] established the concept of an orthogonally α-almost contraction of type D in a b-metric space, as introduced by Istrățescu [15], as follows:

Definition 16 Consider an orthogonal convex b-metric space $(X, \perp, d)$ with a function $\alpha : X \times X \to [0, \infty)$. Originally, an α-almost Istrățescu contraction of type D is a mapping $\Upsilon : X \to X$ if there exist $\tau \in [0,1), \beta \geq 0$ such that for any $x, y \in X$ with $x \perp y$

$$\alpha(x,y) d(\Upsilon^2 x, \Upsilon^2 y) \leq \tau D(x,y) + \beta N(x,y), \tag{8}$$

where

$$D(x,y) = d(\Upsilon x, \Upsilon y) + |d(\Upsilon x, \Upsilon^2 x) - d(\Upsilon y, \Upsilon^2 y)|,$$

and

$$N(x,y) = \min\{d(x,\Upsilon x), d(y,\Upsilon y), d(x,\Upsilon y), d(y,\Upsilon x), d(\Upsilon x,\Upsilon^2 y), d(\Upsilon y,\Upsilon^2 x)\}.$$

Using the above definition, Gnanaprakasam *et al.* [12] proved the following theorem.

Theorem 7 *Assume that* $(X,\perp,d)$ *is an orthogonal convex b-metric space,* $\Upsilon: X \to X$ *is an orthogonal* α*-almost Istrățescu contraction of type D, and* $\alpha: X\times X \to [0,\alpha)$ *has the following properties*:

(i) Υ *is orthogonality-preserving*;
(ii) *for any* $z,\theta \in X$, $\alpha(z,\theta)\geq 1$ *with* $z\perp\theta$, *where* $z\in Fix(X): \Upsilon z = z$;
(iii) Υ *is orthogonally continuous*;
(iv) Υ^2 *is orthogonally continuous with* $\Upsilon^z \perp z$ *and* $\alpha(z,z)\geq 1$, *for any* $z\in X$.

If Υ *is an orthogonal* α*-O-set and there exists* $x_0\in X$ *such that* $x_0\perp\Upsilon x_0$ *and* $\alpha(x_0,\Upsilon x_0)\geq 1$, *then* Υ *has a unique fixed point.*

Aydi *et al.* [3] presented the definition as follows.

Definition 17 Assume that $\mathbb{N}$ denotes the set of positive integers and ψ represents the set of functions that satisfy:

(i) ψ is upper semicontinuous;
(ii) for all $t>0$, $\psi(t)<t$;
(iii) the series $\sum_{n=1}^{\infty}\psi^n(t)<\infty$ for each $t\in\mathbb{R}^+$, where ψ^n is the *nth* iteration of ψ.

The results of Karapinar [19] on interpolative mappings in metric spaces were extended by Nazam *et al.* [28], who established the results based on orthogonal ψ-ϕ interpolative contraction in metric spaces.

Theorem 8 *Let* $(\Upsilon,\perp,d)$ *be an orthogonal metric space, and let* $\psi,\phi:(0,\infty)\to(0,\infty)$ *be two functions. A mapping* $\Upsilon: X\to X$ *is called a* (ϕ,ψ)*-orthogonal Kannan-type contraction if there exists* $v\in(0,1)$ *such that*

$$\psi(d(\Upsilon x,\Upsilon y)) \leq \phi([d(x,\Upsilon x)]^{\nu}.[d(y,\Upsilon y)]^{1-\nu}), \tag{9}$$

for all $x,y\in X$, *and*

$$\min\{d(\Upsilon x,\Upsilon y), d(y,\Upsilon y), d(x,\Upsilon x)\} > 0.$$

3 Main results

This section begins by establishing a fixed point theorem for orthogonal convex interpolative relational mappings in convex cone metric spaces that are TVS valued.

Theorem 9 *Assume that* $(X,\perp,d)$ *is an orthogonal convex complete relational TVS-valued cone metric space over a solid cone P. Let* $\Upsilon: X\to X$ *be an orthogonal interpolative nearly Istrățescu contraction of type D, satisfying the following conditions*:

(i) Υ *is orthogonality-preserving*;
(ii) *for any* $z,\theta\in X$ *and* $z\perp\theta$, *where* $z\in Fix(X): \Upsilon z = z$;

(iii) Υ *is orthogonally continuous if there exists* $\tau \in [0,1), \zeta \leq 1$ *such that for any* $x, y \in X$ *with* $x \perp y$

$$d(\Upsilon^2 x, \Upsilon^2 y) \preceq \tau [D(x,y)]^{\zeta}.[N(x,y)]^{1-\zeta}, \tag{10}$$

where

$$D(x,y) = d(\Upsilon x, \Upsilon y) + \left|\frac{d(\Upsilon x, \Upsilon^2 x) + d(\Upsilon y, \Upsilon^2 y)}{2}\right|,$$

and

$$N(x,y) = \min\{d(x,\Upsilon x), d(y,\Upsilon y), d(x,\Upsilon^2 y), d(y,\Upsilon^2 y), \\ d(\Upsilon y, \Upsilon^2 y), d(\Upsilon y, \Upsilon x)\};$$

(iv) Υ^2 *is orthogonal continuous with* $\Upsilon z \perp z$ *for any* $z \in X$.

If Υ *is an orthogonal O-set and there exists* $x_0 \in X$ *such that* $x_0 \perp \Upsilon x_0$, *then* Υ *has a unique approximate fixed point property.*

Proof Let x_0 be an arbitrary point in X. By definition of orthogonality, we find that

$$x_n \perp \Upsilon x_n \vee \Upsilon x_n \perp x_n$$

for all $n \in \mathbb{N} \cup \{0\}$.

Let $x_n = \Upsilon^n x_0$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_n = \Upsilon^n x_0 = \Upsilon^{n+1} x_0 = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $\Upsilon^n x_0 = x_n$ is a fixed point of Υ, and thus the proof is complete. Otherwise, $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, we claim that $d(\Upsilon x_n, \Upsilon x_{n+1}) > 0$. Since Υ is $\perp$ an orthogonal preserving, we have

$$\Upsilon^n x_0 \perp \Upsilon^{n+1} x_0 \vee \Upsilon^{n+1} x_0 \perp \Upsilon^n x_0$$

for all $n \in \mathbb{N} \cup \{0\}$, which implies that $\{\Upsilon^n x_0\}$ is an orthogonal sequence. Since Υ is an orthogonally interpolative convex almost Istrăţescu contraction of type D, we have $d(\Upsilon x_0, \Upsilon^2 x_0) > 0$.

Taking $x = x_0$ and $y = \Upsilon x_0$ in inequality (10), we obtain

$$d(\Upsilon^2 x_0, \Upsilon^3 x_0) \preceq \tau [D(x_0, \Upsilon x_0)]^{\zeta}.[N(x_0, \Upsilon x_0)]^{1-\zeta}, \tag{11}$$

where

$$D(x_0, \Upsilon x_0) = d(\Upsilon x_0, \Upsilon^2 x_0) + \left|\frac{d(\Upsilon x_0, \Upsilon^2 x_0) + d(\Upsilon^2 x_0, \Upsilon^3 x_0)}{2}\right|.$$

Letting

$$d(\Upsilon x_0, \Upsilon^2 x_0) \preceq \left|\frac{d(\Upsilon x_0, \Upsilon^2 x_0) + d(\Upsilon^2 x_0, \Upsilon^3 x_0)}{2}\right|,$$

we have

$$D(x_0,\Upsilon x_0)=2d(\Upsilon x_0,\Upsilon^2x_0), \tag{12}$$

and

$$\begin{aligned}N(x_0,\Upsilon x_0) &= \min\{d(x_0,\Upsilon x_0),d(\Upsilon x_0,\Upsilon^2x_0),d(x_0,\Upsilon^3x_0),d(\Upsilon x_0,\Upsilon^3x_0),\\ &\quad d(\Upsilon^2x_0,\Upsilon^3x_0),d(\Upsilon^2x_0,\Upsilon x_0)\}.\\ &\preceq \min\{d(x_0,\Upsilon x_0),d(\Upsilon x_0,\Upsilon^2x_0),\\ &\quad [d(x_0,\Upsilon x_0)+d(\Upsilon x_0,\Upsilon^2x_0)+d(\Upsilon^2x_0,\Upsilon^3x_0)],\\ &\quad [d(\Upsilon x_0,\Upsilon^2x_0)+d(\Upsilon^2x_0,\Upsilon^3x_0)],d(\Upsilon^2x_0,\Upsilon^3x_0),\\ &\quad d(\Upsilon^2x_0,\Upsilon x_0)\},\\ &= d(\Upsilon^2x_0,\Upsilon^3x_0). \end{aligned} \tag{13}$$

Using equations (12) and (13) in inequality (11), we get

$$\begin{aligned} d(\Upsilon^2x_0,\Upsilon^3x_0) &\preceq \tau[2d(\Upsilon x_0,\Upsilon^2x_0)]^{\zeta}.[d(\Upsilon^2x_0,\Upsilon^3x_0)]^{1-\zeta},\\ &\preceq 2^{\zeta}\tau[d(\Upsilon x_0,\Upsilon^2x_0)]^{\zeta}.[d(\Upsilon^2x_0,\Upsilon^3x_0)]^{1-\zeta},\\ \Big[d(\Upsilon^2x_0,\Upsilon^3x_0)\Big]^{1-(1-\zeta)} &\preceq 2^{\zeta}\tau[d(\Upsilon x_0,\Upsilon^2x_0)]^{\zeta},\\ \Big[d(\Upsilon^2x_0,\Upsilon^3x_0)\Big]^{\zeta} &\preceq 2^{\zeta}\tau[d(\Upsilon x_0,\Upsilon^2x_0)]^{\zeta},\\ d(\Upsilon^2x_0,\Upsilon^3x_0) &\preceq [2^{\zeta}\tau]^{\frac{1}{\zeta}}d(\Upsilon x_0,\Upsilon^2x_0),\\ d(\Upsilon^2x_0,\Upsilon^3x_0) &\preceq 2\tau^{\frac{1}{\zeta}}d(\Upsilon x_0,\Upsilon^2x_0). \end{aligned}$$

By repeating this process several times, we get the following result:

$$d(\Upsilon^2x_0,\Upsilon^3x_0)\preceq 2^n\tau^{\frac{n}{\zeta}}d(x_0,\Upsilon x_0).$$

Consequently, we have

$$d(\Upsilon^nx_0,\Upsilon^{n+1}x_0)\preceq 0,$$

as $n\to\infty$, which is a contradiction. Hence, $\{\Upsilon^nx_0\}$ is an orthogonal Cauchy sequence.

If $m,n\in\mathbb{N}$, $n<m$, and $m=n+1$, we have

$$\begin{aligned} d(\Upsilon^nx_0,\Upsilon^mx_0) &\preceq d(\Upsilon^nx_0,\Upsilon^{n+1}x_0)+d(\Upsilon^{n+1}x_0,\Upsilon^{n+2}x_0)+\\ &\quad \cdots+d(\Upsilon^{m-1}x_0,\Upsilon^mx_0),\\ &\preceq 2^n\tau^{\frac{n}{\zeta}}d(x_0,\Upsilon x_0)+2^{n+1}\tau^{\frac{n+1}{\zeta}}d(x_0,\Upsilon x_0)+\\ &\quad \cdots+2^{m-1}\tau^{\frac{m-1}{\zeta}}d(x_0,\Upsilon x_0),\\ &\preceq [2^n\tau^{\frac{n}{\zeta}}+2^{n+1}\tau^{\frac{n+1}{\zeta}}+\cdots+2^{m-1}\tau^{\frac{m-1}{\zeta}}]d(x_0,\Upsilon x_0), \end{aligned}$$

$$\begin{aligned}
&\preceq 2^n \tau^{\frac{n}{\zeta}}[1 + 2^1\tau^{\frac{1}{\zeta}} + \cdots + 2^{m-n-1}\tau^{\frac{m-n-1}{\zeta}}]d(x_0, \Upsilon x_0),\\
&\preceq \frac{2^n \tau^{\frac{n}{\zeta}}}{1 - 2\tau^{\frac{1}{\zeta}}} d(x_0, \Upsilon x_0),
\end{aligned}$$

as $n \to \infty$, we have $d(\Upsilon^n x_0, \Upsilon^m x_0) \preceq 0$. Therefore, $\{\Upsilon^n x_0\}$ is an orthogonal Cauchy sequence.

Since $(X, \perp, d)$ is a complete an $\perp$-TVS-valued cone metric space and Υ^2 is $\perp$-continuous, we have $\Upsilon^2 z = z$, that is z forms a fixed point of Υ. Let $\Upsilon z \neq z$, and let $x = x_0 = x_{n_k}$ and $y = \Upsilon x_0 = z$ for $k \in \mathbb{N} \cup \{0\}$ and $\lim\limits_{k\to\infty} x_{n_k} = z$. Using inequality (10), we have

$$d(\Upsilon^2 x_{n_k}, \Upsilon^2 z) \preceq \tau[D(x_{n_k}, z)]^{\zeta}.[N(x_{n_k}, z)]^{1-\zeta}, \tag{14}$$

where

$$\begin{aligned}
D(x_{n_k}, z) &= d(\Upsilon x_{n_k}, \Upsilon z) + \left|\frac{d(\Upsilon x_{n_k}, \Upsilon^2 x_{n_k}) + d(\Upsilon z, \Upsilon^2 z)}{2}\right|,\\
&\preceq d(\Upsilon z, \Upsilon z) + \left|\frac{d(\Upsilon z, \Upsilon^2 z) + d(\Upsilon z, \Upsilon^2 z)}{2}\right|,\\
&= d(\Upsilon z, \Upsilon^2 z),
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
N(x_{n_k}, z) &= \min\{d(x_{n_k}, \Upsilon x_{n_k}), d(z, \Upsilon z), d(x_{n_k}, \Upsilon^2 z), d(z, \Upsilon^2 z),\\
&\quad d(\Upsilon z, \Upsilon^2 z), d(\Upsilon z, \Upsilon x_{n_k})\},\\
&\preceq \min\{d(z, \Upsilon z), d(z, \Upsilon z), d(z, \Upsilon^2 z), d(z, \Upsilon^2 z),\\
&\quad d(\Upsilon z, \Upsilon^2 z), d(\Upsilon z, \Upsilon z)\},\\
&= d(\Upsilon z, \Upsilon^2 z).
\end{aligned} \tag{16}$$

From inequalities (15) and (16) in (14), we get

$$\begin{aligned}
d(\Upsilon z, \Upsilon^2 z) &\preceq \tau[d(\Upsilon z, \Upsilon^2 z)]^{\zeta}.[d(\Upsilon z, \Upsilon^2 z)]^{1-\zeta},\\
&\preceq \tau d(\Upsilon z, \Upsilon^2 z),\\
(1-\tau)d(\Upsilon z, \Upsilon^2 z) &\preceq 0,\\
d(\Upsilon z, \Upsilon^2 z) &\preceq 0,
\end{aligned}$$

which implies that

$$d(\Upsilon z, \Upsilon^2 z) = 0.$$

Thus, $\Upsilon z = \Upsilon^2 z$. This shows that $\Upsilon z = z$, and hence z is a fixed point of Υ in X.

For uniqueness of z, let $z, \theta \in X$ and $z \perp \theta$, where $z \in Fix(X) : \Upsilon z = z$. Also $\theta \in Fix(X) : \Upsilon\theta = \theta$. Assume that $z \neq \theta$ and $\Upsilon z \neq \Upsilon\theta$, let $x = z$ and $y = \theta$, using inequality (10) gives

$$d(\Upsilon^2 z, \Upsilon^2 \theta) \preceq \tau[D(z, \theta)]^{\zeta}.[N(z, \theta)]^{1-\zeta}, \tag{17}$$

where

$$\begin{aligned} D(z,\theta) &= d(\Upsilon z, \Upsilon\theta) + \left|\frac{d(\Upsilon z, \Upsilon^2\theta) + d(\Upsilon\theta, \Upsilon^2\theta)}{2}\right|, \\ &\preceq d(\Upsilon z, \Upsilon\theta) + \left|\frac{d(\Upsilon z, \Upsilon^2\theta)}{2}\right|, \\ &\preceq \frac{2d(\Upsilon z, \Upsilon\theta) + d(\Upsilon z, \Upsilon\theta)}{2}, \\ &= \frac{3d(\Upsilon z, \Upsilon\theta)}{2}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} N(z,\theta) &= \min\{d(z,\Upsilon z), d(\theta,\Upsilon\theta), d(z,\Upsilon^2\theta), d(\theta,\Upsilon^2\theta), \\ &\quad d(\Upsilon\theta,\Upsilon^2\theta), d(\Upsilon\theta,\Upsilon z)\}, \\ &\preceq \min\{0,0,d(z,\Upsilon^2\theta),0,0,d(\Upsilon\theta,\Upsilon z)\}, \\ &= 0. \end{aligned}$$

Applying all of the above equalities, we obtain

$$\begin{aligned} &d(\Upsilon^2 z, \Upsilon^2\theta) \preceq \tau\left[\frac{3d(\Upsilon z, \Upsilon\theta)}{2}\right]^{\zeta}.[0]^{1-\zeta}, \\ &d(\Upsilon^2 z, \Upsilon^2\theta) \preceq 0. \end{aligned}$$

This implies that $\Upsilon^2 z = \Upsilon^2\theta$, that is $z = \theta$. Therefore, z is a unique fixed point of Υ. This completes the proof. □

Example 6 The relation $\perp$ on X is defined by the set of real numbers $X = \mathbb{R}^2$. Using the formula $P = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$, define the cone $P \subseteq \mathbb{R}^2$ so that $x \perp y$ if x and y fulfill $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Let and $\perp = \{(-2,0),(2,0),(0,-3),(0,3)\}$ be TVS-values convex cone metric spaces that are orthogonal. Define a TVS-valued cone metric with a Frobenius norm by

$$d(x,y) = \|Q_1 - Q_2\|_F = \sqrt{\sum_{i,j} |(Q_1 - Q_2)_{ij}|^2 \psi(t)},$$

where $\psi(t) = e^t$. Define the distance between two orthogonal matrices and consider a mapping $\Upsilon : X \to X$ given by

$$\Upsilon(x,y) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which represents the rotation matrix by 45 degrees.

To calculate the distance between two orthogonal matrices $x \perp y \vee Q_1 \perp Q_2$ using the Frobenius norm, one should follow these steps:

Let $x = (x_1, y_1) = (2,0)$, $y = (x_2, y_2) = (0,3)$ and $\zeta = 0.25$. Using inequality (10), we get

$$d(\Upsilon^2 x, \Upsilon^2 y) = \left\| \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\|,$$

$$
\begin{aligned}
&= \left\| \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\|, \\
&= \left\| \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\|, \\
&= \left\| \begin{bmatrix} -3 \\ -2 \end{bmatrix} \right\|, \\
&= \sqrt{(-3)^2 + (-2)^2}e^t = \sqrt{9 + 4}e^t = \sqrt{13}e^t.
\end{aligned}
$$

Similarly, we obtain the following Frobenius norm TVS-valued cone metrics

$$
\begin{aligned}
d(\Upsilon x, \Upsilon y) &= \left\| \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} \right\|, \\
&= \left\| \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-5}{\sqrt{2}} \end{bmatrix} \right\|, \\
&= \sqrt{(\frac{-1}{\sqrt{2}})^2 + (\frac{-5}{\sqrt{2}})^2}e^t = \sqrt{\frac{1}{2} + \frac{25}{2}}e^t = \sqrt{13}e^t. \\
d(\Upsilon x, \Upsilon^2 x) &= \left\| \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{2\sqrt{2}-2}{\sqrt{2}} \end{bmatrix} \right\|, \\
&= \sqrt{\left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{2\sqrt{2}-2}{\sqrt{2}}\right)^2}e^t = \sqrt{8 - 4\sqrt{2}}e^t. \\
d(\Upsilon y, \Upsilon^2 y) &= \left\| \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{3-3\sqrt{2}}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix} \right\|, \\
&= \sqrt{\left(\frac{3-3\sqrt{2}}{\sqrt{2}}\right)^2 + \left(\frac{3}{\sqrt{2}}\right)^2}e^t = \sqrt{18 - 9\sqrt{2}}e^t. \\
d(\Upsilon x, \Upsilon^2 y) &= \left\| \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{2-3\sqrt{2}}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{bmatrix} \right\|, \\
&= \sqrt{\left(\frac{2-3\sqrt{2}}{\sqrt{2}}\right)^2 + \left(\frac{-2}{\sqrt{2}}\right)^2}e^t = \sqrt{13 - 6\sqrt{2}}e^t, \\
d(y, \Upsilon^2 y) &= \left\| \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ -3 \end{bmatrix} \right\|. \\
&= \sqrt{(3)^2 + (-3)^2}e^t = \sqrt{18}e^t. \\
d(y, \Upsilon y) &= \left\| \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{3}{\sqrt{2}} \\ \frac{-3}{\sqrt{2}} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{3\sqrt{2}-3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix} \right\|, \\
&= \sqrt{\left(\frac{3\sqrt{2}-3}{\sqrt{2}}\right)^2 + \left(\frac{3}{\sqrt{2}}\right)^2}e^t = \sqrt{18 - 9\sqrt{2}}e^t. \\
d(x, \Upsilon x) &= \left\| \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{-2}{\sqrt{2}} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{2\sqrt{2}-2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} \right\|,
\end{aligned}
$$

$$= \sqrt{\left(\frac{2\sqrt{2}-2}{\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2 e^t} = \sqrt{8-4\sqrt{2}e^t}.$$

Using all of the above equations in (10), we obtain

$$\sqrt{13}e^t \preceq \tau[D(x,y)]^\zeta.[N(x,y)]^{1-\zeta}, \tag{19}$$

where

$$\begin{aligned} D(x,y) &= \sqrt{13}e^t + \left|\frac{\sqrt{8-4\sqrt{2}}e^t + \sqrt{18}e^t}{2}\right|, \\ &= 3.6e^t + \left|\frac{1.5e^t + 4.24e^t}{2}\right| = 6.47e^t, \end{aligned}$$

and

$$\begin{aligned} N(x,y) = \min &\left\{\sqrt{8-4\sqrt{2}}e^t, \sqrt{18-9\sqrt{2}}e^t, \sqrt{13-6\sqrt{2}}e^t, \sqrt{18}e^t,\right. \\ &\left.\sqrt{18}e^t, \sqrt{13}e^t\right\}, \\ = \min &\left\{1.5e^t, 2.96e^t, 2.125e^t, 4.24e^t, 4.24e^t, 3.6e^t\right\} = 1.5e^t. \end{aligned}$$

It also implies that

$$\begin{aligned} &3.6e^t \preceq \tau[6.47e^t]^\zeta.[1.5e^t]^{1-\zeta}, \\ &3.6e^t \preceq \tau e^t[6.47]^\zeta.[1.5]^{1-\zeta}. \end{aligned}$$

Letting $\tau = 0.95, t = 2, \zeta = 0.667$, we have

$$\begin{aligned} &3.6e^t \preceq \tau e^t[6.47]^\zeta.[1.5]^{1-\zeta}, \\ &\quad 3.6 \preceq \tau[6.47]^{0.667}.[1.5]^{0.333}. \\ &\quad 3.6 \preceq 0.95 \times 3.474 \times 1.1446, \\ &\quad 3.6 \preceq 3.777. \end{aligned}$$

This demonstrates Theorem 9. Therefore, all imposed conditions are satisfied. Consequently, this example cannot be solved by existing results, which underscores the significance of our findings.

To verify this claim, consider the following contraction:

(1) Taking $d(\Upsilon x, \Upsilon y) = \sqrt{13}e^t = 3.6e^t, d(x, \Upsilon x) = 2.3e^t, d(y, \Upsilon y) = 1.53e^t$ and $\nu = 0.9$ in (9), we get

$$\psi(3.6e^t) \leq \phi(2.26e^t),$$

for the continuity property of ψ and ϕ this concludes that

$$3.6 \geq 2.26,$$

which is a contradiction.

(2) Applying $d(\Upsilon^2 x, \Upsilon^2 y) = 3.6e^t, D(x,y) = 4.37e^t, N(x,y) = 1.53e^t$ and $\tau = 0.25, \beta = 0, \alpha(x,y) = 1$ in (8), we obtain

$$3.6e^t \leq 0.25 \times 4.37e^t + 0 \times 1.53e^t,$$

$$3.6e^t \leq 1.09e^t,$$

implying that

$$3.6 \geq 1.09,$$

which is a contradiction.

This concludes that Υ is an orthogonal interpolative almost Istrățescu contraction of type D, not orthogonal almost Istrățescu contraction of type D or Istrățescu contraction.

In the following results, we extend Theorem 8 by utilizing interpolative Chartejea-Reich-Rus-Ćirić-Kannan-type contractions within an orthogonal TVS-valued cone metric space.

Theorem 10 *Suppose* $(\Upsilon, \perp, d)$ *is an orthogonal cone metric space with TVS values, and* $\psi : (0,\infty) \to (0,\infty)$ *is a function. A* (ϕ, ψ)*-orthogonal, interpolative Chartejea-Reich-Rus-Ćirić-Kannan-type contraction is defined as a mapping* $\Upsilon : X \to X$ *if there exists* $\tau \in (0,1)$ *such that*

$$d(\Upsilon x, \Upsilon y) \preceq \psi(\tau \mathcal{M}_S(x,y)), \tag{20}$$

where

$$\mathcal{M}_S(x,y) = [d(x,y)]^{\gamma_1}, \Big[\max\{d(x,y), d(x,\Upsilon y), d(y,\Upsilon x)\}\Big]^{\gamma_2}. \Big[\frac{d(x,\Upsilon x).d(y,\Upsilon y)}{d(x,y)}\Big]^{1-\gamma_1-\gamma_2}, \tag{21}$$

$\forall\, x \in X$ *with* $x, y \notin Fix(\Upsilon)$ *and* $(x,y) \in \perp$. *Then, there exists* $x \in X$ *such that* $x \in \Gamma x$ *with* $\sum_{i=1}^{2} \gamma_i \leq 1$. *Moreover,*

(v) $\Gamma(X)$ *is* $\perp$*-connected or* $\perp$*-preserved. Then,* Υ *have a unique fixed point.*

Proof Consider $(X, \perp)$ is an orthogonal set, there exists

$$x_0 \in X : \forall\, x \in X, x \perp x_0 \vee x \in X, x_0 \perp x.$$

It follows that

$$x_0 \perp \Upsilon x_0 \vee \Upsilon x_0 \perp x_0.$$

Let

$$x_1 = \Upsilon x_0$$

$$
\begin{aligned}
x_2 &= \Upsilon x_1 = \Upsilon^2 x_0 \\
&\quad \ldots\ldots \\
x_n &= \Upsilon x_{n-1} = \Upsilon^n x_0 \ \forall\, n \in \mathbb{N}.
\end{aligned}
$$

Assign $x_n = \Upsilon x_{n-1}$ to $x_0 \in X$. If $n \in \mathbb{N} \cup \{0\}$ exists such that $x_n = x_{n+1}$, then $\Upsilon x_n = x_n$ must exist. Since x_n is a fixed point of Υ, we determine this. Consequently, the proof is finished. In every other case, if $x_n \leq x_{n+1}$, then for any $n \in \mathbb{N} \cup \{0\}$, we have $d(x_n, x_{n+1}) > 0$. Considering that Υ is $\perp$-preserving, we get

$$
x_n \perp x_{n+1} \vee x_{n+1} \perp x_n.
$$

This implies that $\{x_n\}$ is an orthogonal sequence. Let $x = x_{n-1}$ and $y = x_n$. Using inequality (20), we get

$$
d(\Upsilon x_{n-1}, \Upsilon x_n) \preceq \psi(\tau \mathcal{M}_S(x_{n-1}, x_n)), \tag{22}
$$

where

$$
\begin{aligned}
\mathcal{M}_S(x_{n-1}, x_n) &= [d(x_{n-1}, x_n)]^{\gamma_1}.[\max\{d(x_{n-1}, x_n), d(x_{n-1}, \Upsilon x_n), \\
&\qquad d(x_n, \Upsilon x_{n-1})\}]^{\gamma_2}.\Big[\frac{d(x_{n-1}, \Upsilon x_{n-1}).d(x_n, \Upsilon x_n)}{d(x_{n-1}, x_n)}\Big]^{1-\gamma_1-\gamma_2}, \\
&\preceq [d(x_{n-1}, x_n)]^{\gamma_1}.[\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), \\
&\qquad d(x_n, x_n)\}]^{\gamma_2}.\Big[\frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}\Big]^{1-\gamma_1-\gamma_2}, \\
&\preceq [d(x_{n-1}, x_n)]^{\gamma_1}.[d(x_{n-1}, x_n)]^{\gamma_2}[d(x_n, x_{n+1})]^{1-\gamma_1-\gamma_2}, \\
&= [d(x_{n-1}, x_n)]^{\gamma_1+\gamma_2}.[d(x_n, x_{n+1})]^{1-\gamma_1-\gamma_2}.
\end{aligned} \tag{23}
$$

Applying equation (23) in inequality (22), we have

$$
d(x_n, x_{n+1}) \preceq \psi(\tau [d(x_{n-1}, x_n)]^{\gamma_1+\gamma_2}.[d(x_n, x_{n+1})]^{1-\gamma_1-\gamma_2}).
$$

Using nondecreasing property of ψ, we obtain

$$
\begin{aligned}
d(x_n, x_{n+1}) &\prec \tau [d(x_{n-1}, x_n)]^{\gamma_1+\gamma_2}.[d(x_n, x_{n+1})]^{1-\gamma_1-\gamma_2}, \\
\Big[d(x_n, x_{n+1})\Big]^{1-(1-\gamma_1-\gamma_2)} &\prec \tau [d(x_{n-1}, x_n)]^{\gamma_1+\gamma_2}, \\
\Big[d(x_n, x_{n+1})\Big]^{\gamma_1+\gamma_2} &\prec \tau [d(x_{n-1}, x_n)]^{\gamma_1+\gamma_2}, \\
d(x_n, x_{n+1}) &\prec \tau^{\frac{1}{\gamma_1+\gamma_2}} d(x_{n-1}, x_n).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &\preceq \psi(\tau^{\frac{1}{\gamma_1+\gamma_2}} d(x_{n-1}, x_n)), \\
&\preceq \psi^2(\tau^{\frac{2}{\gamma_1+\gamma_2}} d(x_{n-2}, x_{n-1})),
\end{aligned}
$$

$$\begin{aligned}
&\dots \\
&\preceq \psi^n(\tau^{\frac{n}{\gamma_1+\gamma_2}} d(x_0,x_1)),
\end{aligned} \tag{24}$$

as $n \to \infty$, which implies $d(x_n,x_{n+1}) = 0$. Hence, $\{x_n\}$ is an orthogonal Cauchy sequence.

The remaining steps of this proof follow a similar approach to those in Theorem 9. Thus, this completes our proof. □

The following example is used to demonstrate Theorem 10.

Example 7 The set of real numbers that define the relation $\perp$ on X is $X = \mathbb{R}^2$. The cone $P \subseteq \mathbb{R}^2$ can be defined as $P = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$ such that $x \perp y$ if x and y satisfy $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $\perp = \{(0,0),(0,1),(1,0),(1,1)\}$ are complete orthogonal relational TVS-values convex cone metric spaces. Define a cone metric with a Frobenius norm and TVS value by

$$d(x,y) = \|Q_1 - Q_2\|_F = \sqrt{\sum_{i,j} |(Q_1 - Q_2)_{ij}|^2 \psi(t)},$$

where $\psi(t) = e^t$. Define the distance between two orthogonal matrices and consider a mapping $\Upsilon : X \to X$ given by

$$\Upsilon(x,y) = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which represents the orthogonal matrix.

To calculate the distance between two orthogonal matrices $x \perp y \vee Q_1 \perp Q_2$ using the Frobenius norm, one should follow these steps:

Let $x = (x_1,y_1) = (0,1)$, $y = (x_2,y_2) = (1,1)$ and $\gamma_1 = 0.5, \gamma_2 = 0.2, \tau = 0.25$. Using inequality (20), we obtain the following Frobenius norm TVs-valued cone metrics

$$\begin{aligned}
d(\Upsilon x, \Upsilon y) &= e^t, \\
d(x,y) &= e^t, \\
d(x, \Upsilon y) &= 0.447e^t, \\
d(y, \Upsilon x) &= 2.4e^t, \\
d(x, \Upsilon x) &= 1.897e^t, \\
d(y, \Upsilon y) &= 1.265e^t.
\end{aligned}$$

Using all of the above equations in (20), we obtain

$$e^t \preceq \psi(\tau \mathcal{M}_S(x,y)), \tag{25}$$

where

$$\mathcal{M}_S(x,y) = [e^t]^{0.25}, \Big[\max\{e^t, 0.447e^t, 2.4e^t\}\Big]^{0.2} . \Big[\frac{1.897e^t.1.265e^t}{e^t}\Big]^{0.55},$$

$$\begin{aligned}
&\preceq [e^t]^{0.25}, \Big[2.4e^t\Big]^{0.2}.\Big[2.3997\Big]^{0.55}, \\
&\preceq e^{0.45t}, \Big[2.4\Big]^{0.2}.\Big[2.3997\Big]^{0.55}, \\
&= 1.928164e^{0.45t}. \qquad (26)
\end{aligned}$$

Using (26) in (25) yields

$$\begin{aligned}
&e^t \preceq \psi(0.5 \times 1.928164e^{0.45t}), \\
&e^t \preceq e^{0.96e^{0.45t}}.
\end{aligned}$$

For any value of $t = 0, 1, 2 \ldots$, the above inequality is satisfied. This verifies Theorem 10. Therefore, all imposed conditions are satisfied.

Further, we prove the following theorem by extending Theorem 4 using interpolative Hardy-Rogers-type contractions within an orthogonal TVS-valued cone metric space.

Theorem 11 *Let $(X, \perp, d)$ be an orthogonal topological vector space (TVS)-valued cone convex metric space over a solid cone P. Let $\Upsilon : X \to X$ be an interpolative Hardy-Rogers cone convex contraction mapping of order 2 if the following conditions hold*:

(i) *$X(\Upsilon, \perp)$ is non-empty*;

(ii) *$\perp$ is Υ-preserved*;

(iii) *either Υ is $\perp$-continuous, or $(X, \perp, d)$ is asymptotically regular*;

(iv) *there exist $\mu_1, \mu_2, \mu_3 \in [0, 1]$ and $\gamma \geq 2$ such that*

$$\begin{aligned}
d(\Upsilon^2 x, \Upsilon^2 y) \leq \gamma [d(x, \Upsilon x)]^{\mu_1}.[d(\Upsilon x, \Upsilon^2 x)]^{\mu_2}.[d(y, \Upsilon y)]^{\mu_3}. \\
[d(\Upsilon y, \Upsilon^2 y)]^{1-\mu_1-\mu_2-\mu_3}, \qquad (27)
\end{aligned}$$

for all $x, y \in X$, and $\mu_1 + \mu_2 + \mu_3 < 1$, with $x, y \notin Fix(\Upsilon)$ and $(x, y) \in \perp$. Then, Υ has the approximate fixed point property in X, and for any $x \in X$, the iterative sequence $\{\Upsilon^n x\}$ converges to the fixed point. Moreover,

(v) *$\Upsilon(X)$ is $\perp$-connected. Then, Υ has a unique fixed point.*

Proof The proof of this theorem follows a similar argument to that of Theorem 9. This completes the proofs. □

The following example is used to support Theorem 11.

Example 8 Set $X = A = \mathbb{R}^2$, where $\perp$ on X is defined by the set of real numbers. Define the cone $P \subseteq \mathbb{R}^2$ using $P = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$ such that $x \perp y$ if x and y fulfill $\{1, 2, 3\}$, and $\perp = \{(1, 2), (2, 3), (2, 1), (3, 1), (3, 2)\}$ are complete orthogonal relational TVS-values convex cone metric spaces. A TVS-valued cone metric with a Frobenius norm is defined by

$$d(x, y) = \|Q_1 - Q_2\|_F = \sqrt{\sum_{i,j} |(Q_1 - Q_2)_{ij}|^2 \psi(t)},$$

where $\psi(t) = e^t$. Define the distance between two orthogonal matrices and consider a mapping $\Upsilon : X \to X$ given by

$$\Upsilon(x,y) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which represents the rotation matrix by $\theta = \frac{2\pi}{n}$ degrees, for $n > 0$.

We used the concept of Frobenius norm in a TVS-valued metric to calculate the distance between two matrices.

A set of orthogonal binary relations is a collection of binary relations on a set such that any two relations in the set are pairwise orthogonal. Orthogonality of relations means they do not share any common pairs, i.e., the intersections of their graphs are empty.

Let $A = \{1,2,3\}$. We define three binary relations R_1, R_2, and R_3 on A.

$R_1 = \{(1,2),(2,3)\}$

$R_2 = \{(2,1),(3,1)\}$

$R_3 = \{(3,2)\}$

Two relations R_i and R_j (where $i \neq j$) are orthogonal if

$$R_i \cap R_j = \emptyset$$

1. $R_1 \cap R_2$:

$$R_1 = \{(1,2),(2,3)\}, \quad R_2 = \{(2,1),(3,1)\}.$$

There are no common pairs, so $R_1 \cap R_2 = \emptyset$.

2. $R_1 \cap R_3$:

$$R_1 = \{(1,2),(2,3)\}, \quad R_3 = \{(3,2)\}.$$

There are no common pairs, so $R_1 \cap R_3 = \emptyset$.

3. $R_2 \cap R_3$:

$$R_2 = \{(2,1),(3,1)\}, \quad R_3 = \{(3,2)\}.$$

There are no common pairs, so $R_2 \cap R_3 = \emptyset$.

To calculate the distance between two orthogonal matrices $x \perp y \vee Q_1 \perp Q_2$ using the Frobenius norm, one should follow these steps:

Let $x \in R_1 = (x_1,y_1) = (1,2)$, $y \in R_2 = (x_2,y_2) = (3,1), n = 6$ and $\mu_1 = 0.2, \mu_2 = 0.1, \mu_3 = 0.3, \gamma = 2$, using inequality (27), we get

$$\begin{aligned} d(\Upsilon^2 x, \Upsilon^2 y) &= \sqrt{5}e^t = 2.24e^t, \\ d(x, \Upsilon x) &= \sqrt{3.2}e^t = 1.788e^t, \\ d(\Upsilon x, \Upsilon^2 x) &= 0.4994e^t, \\ d(y, \Upsilon y) &= 1.64e^t, \\ d(\Upsilon y, \Upsilon^2 y) &= 2.68e^t. \end{aligned}$$

From inequality (32) and all inequalities obtained above, we get

$$
\begin{aligned}
&2.24e^t \leq \gamma[1.788e^t]^{0.2}.[0.4994e^t]^{0.1}.[1.64e^t]^{0.3}.[2.68e^t]^{0.4},\\
&2.24e^t \leq \gamma[1.788]^{0.2}.[0.4994]^{0.1}.[1.64]^{0.3}[2.68]^{0.4}e^t,\\
&\quad 2.24 \leq 1.22 \times 0.9329 \times 1.16 \times 1.483\gamma,\\
&\quad 2.24 \leq 1.98\gamma.
\end{aligned}
$$

This verifies Theorem 11. Therefore, all imposed conditions are satisfied.

According to the Buser concept [9], let $A = \Upsilon$ be a mapping and $b = y$. Using inequality (4), we formulate the following corollary.

corollary 12 *Consider the orthogonal metric space* $(\Upsilon, \perp, d)$ *and the function* $\psi : (0,\infty) \to (0,\infty)$. *A* ψ*-orthogonal interpolative Kannan-type contraction is defined as a mapping* $\Upsilon : X \to X$ *if there is* $\sigma, \zeta \in (0,1)$ *such that*

$$\psi(d(y,\Upsilon y)) \leq \zeta[d(y,\Upsilon x)]^{\sigma}.[d(\Upsilon x,\Upsilon y)]^{1-\sigma}. \tag{28}$$

Then, Υ *has a unique fixed point in* X.

Proof The proof of this corollary follows a similar argument to that of Theorem 10. This completes the proofs. □

4 An application to electric circuit second-order differential equations in orthogonal TVS-valued metric space

This section derives the second-order differential equation for an electric circuit to verify the results of Theorem 9. To do this, we transform the second-order differential equation into a system of integral equations. Fixed point theory, which plays a key role in physics applications, is specifically used to solve electric circuit equations. This approach is inspired by the work of Saipara [34]. It is well known that an electric circuit can be represented by a ternary relation involving a resistor R, an inductor L, and a capacitor C in series with an electromotive force E. Alternatively, the circuit can be represented using ternary relations for inductive reluctance X_L, capacitive reactance X_C, and impedance Z, which are orthogonal to each other. If the rate of charge q in the capacitor with respect to time t is denoted as the current I, such that $I = \frac{dq}{dt}$, the following ternary relations are obtained:

$$
\begin{aligned}
&V = IR,\\
&V = \frac{q}{C},\\
&V = L\frac{dI}{dt}.
\end{aligned}
$$

According to the Kirchhorff law, the sum of voltage drops across the circuit is equal to the supplied voltage. We consider the $R - L - C$ circuit. The differential equation for the charge q on the capacitor is

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t),$$

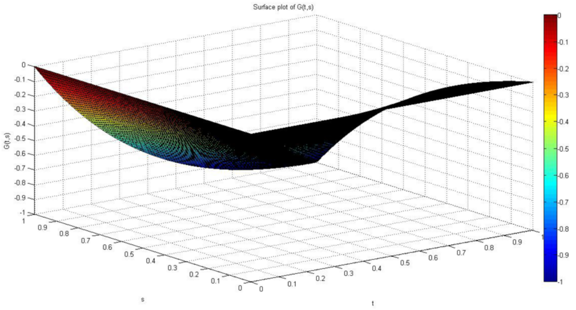

Figure 2 *G*(*t*, *s*)-Graph

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t), \tag{29}$$

where $V(t)$ is the applied voltage at time t. If the initial conditions are assumed to be

$$q(0) = 0, \frac{dq(0)}{dt} = 0\ , \ R^2 > \frac{4L}{C} \ and \ \tau = \frac{R}{2L}.$$

The differential equation (29) can be represented using the Green function given by

$$G(t,s) = \begin{cases} -se^{\theta(s-t)}, 0 \leq s \leq t \leq 1, \\ -te^{\theta(s-t)}, 0 \leq t \leq s \leq 1. \end{cases}$$

The green function can also be represented by the graph in Fig. 2.

The differential equation (29) can be transformed into an integral equation given by

$$x(t) = \int_0^T G(t,s)K(t,s,x(s))ds + g(t), \tag{30}$$

for all $t, s \in X = [0, T], T > 0$, where $K : X \times X \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : X \to \mathbb{R}^n$. Let $\perp$ be a ternary relation on $\mathbb{R}^n$ and $X = C(X, \mathbb{R}^n)$ on X, set of all continuous mappings from $\mathcal{X} \to \mathbb{R}^n$. Consider the binary relation $x \perp y$ on X as

$$(x,y) \in \perp \Leftrightarrow x \perp y \ or \ y \perp x, \tag{31}$$

for all $t \in X$.

Now, equation (29) can be transformed into the following fixed point equation:

$$\Upsilon^2 x(t) = \int_0^T G(t,s)K(t,s,x(s))ds + g(t).$$

for all $t, s \in X$.

Let $X = C([0, T])$ be the set of all continuous function defined on $[0, T]$. Define an orthogonal TVS-valued metric on X, by $d : X \times X \to \mathbb{R}^n$, $\forall\ n\ \geq 2$ and

$$d(x,y) = \sup_{t \in [0,T]} \|x - y\|e^t.$$

The, n (X, d) is a complete TVS-valued cone metric space.

Theorem 13 *Assume that the following conditions hold*:

(i) $K:[0,T]\times[0,T]\times\mathbb{R}\to\mathbb{R}$ *and* $g:\mathbb{R}\to\mathbb{R}$ *are continuous*;

(ii) *there exist some* $t,s\in[0,T]$ *and* $x\in X$ *such that*

$$\int_0^T G(t,s)K(t,s,x(s))ds+g(t),\int_0^T G(t,s)K(t,s,y(s)))ds+g(t)\in X,$$

for all $x\perp y$.

(iii) *there exists some function* $K:[0,T]\times[0,T]\times[0,T]\to[\mathbb{R}^n]$, *for each* $x\perp y\in X$ *or* $y\perp x\in X$ *and* $t,s\in[0,T]$ *with* $\tau\in[0,1]$ *such that*

$$\|K(t,s,x(s))-K(t,s,y(s))\|\le\tau\|x-y\|e^t,$$

where

$$d(x,y)=\|x-y\|e^t=[D(x,y)]^{\zeta}.[N(x,y)]^{1-\zeta},$$

and

$$D(x,y)=d(\Upsilon x,\Upsilon y)+\left|\frac{d(\Upsilon x,\Upsilon^2x)+d(\Upsilon y,\Upsilon^2 y)}{2}\right|,$$

with

$$\begin{aligned}N(x,y)=\min\{&d(x,\Upsilon x),d(y,\Upsilon y),d(x,\Upsilon^2y),d(y,\Upsilon^2y),\\&d(\Upsilon y,\Upsilon^2y),d(\Upsilon y,\Upsilon x)\}.\end{aligned}$$

(iv) *there exists* $\tau\in[0,1)$ *such that*

$$G(t,s)=\frac{1-e^{-\theta t}-\theta te^{\theta(1-t)}}{\theta^2}\le\tau,$$

where $\theta=\frac{R}{2L}$.

Then, the system of Equations (29) *has a unique solution* $x(t)$, *which is a solution to the integral Equation* (30).

Proof For $x,y\in C([0,T])$ with $x\perp y$, we claim that $d(\Upsilon^2x,\Upsilon^2y)\nleq\tau d(x,y)$. Then, we have

$$\begin{aligned}\|\Upsilon^2x-\Upsilon^2y\|&\le\Bigg\|\Big(\int_0^T G(t,s)K(t,s,x(s))ds+g(t)\Big)-\\&\qquad\Big(\int_0^T G(t,s)K(t,s,x(s))ds+g(t)\Big)\Bigg\|,\\&\le\int_0^T G(t,s)ds\|(K(t,s,x(s))-K(t,s,y(s)))\|,\\&\le\sup_{0\in[0,T]}\|x(t)-y(t)\|e^t\int_0^T G(t,s)ds,\\&\le\sup_{0\in[0,T]}\|x(t)-y(t)\|e^t\left[\int_0^t -se^{\theta(s-t)}ds-\int_0^1 te^{\theta(s-t)}ds\right],\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{0\in[0,T]} \|x(t)-y(t)\|e^{t}\left[\frac{1-e^{-\theta t}-\tau t e^{\theta(1-t)}}{\theta^{2}}\right],\\
&\leq \tau \sup_{0\in[0,T]} \|x(t)-y(t)\|e^{t},\\
d(\Upsilon^{2}x, \Upsilon^{2}y) &\leq \tau d(x,y),
\end{aligned}
$$

which is a contradiction. Therefore, a solution to the integral equation (30) and a second-order differential equation (29), x, is a unique fixed point of Υ. As a result, we can say that every requirement set forth in Theorems 9 and 13 is met. Thus, the proof is finished. □

4.1 An application to matrix equation in orthogonal TVS-valued metric space

This subsection demonstrates Theorem 10 with an application to a matrix equation. Consider the following system of linear equations with n unknowns motivated from [30].

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{1n}x_n &= b_3,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= \ldots\\
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{nn}x_n &= b_n.
\end{aligned}
\tag{32}
$$

This system of equations can be transformed into

$$
\begin{aligned}
x_1 &= (1-a_{11})x_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n + b_1\\
x_2 &= -a_{21}x_1 + (1-a_{22})x_2 - a_{23}x_3 - \cdots - a_{2n}x_n + b_2\\
x_3 &= -a_{31}x_1 - a_{32}x_2 + (1-a_{33})x_3 - \cdots - a_{1n}x_n + b_3,\\
\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
x_n &= -a_{n1}x_1 - a_{n2}x_2 - a_{n3}x_3 - \cdots + (1-a_{nn})x_n + b_n
\end{aligned}
$$

Let $\alpha_{ij} = -a_{ij} + \delta_{ij}$, where

$$
\delta_{ij} = \begin{cases} 1, 0\forall\ i = j, \\ 0,\ \forall\ i \neq j. \end{cases}
$$

Then, the system is equivalent to

$$
x_i = \sum_{j=1}^{n} \alpha_{ij}x_j + b_i, i = 1, 2\ldots. \tag{33}
$$

If $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$, $b = (b_1, b_2, \ldots b_n) \in \mathbb{R}^n$ and $A = (a_{ij})_{n\times n}a$ matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix},
$$

and

$$x = (x_1, x_2, \ldots x_n)^T,$$
$$b = (b_1, b_2, \ldots b_n)^T.$$

Equation (33) is equivalent to

$$x = x - Ax + b,$$
$$x = (I - A)x + b,$$
$$x = Ax + b.$$

Now, we prove the following theorem using the concepts of orthogonal TVS-valued cone metric space.

Theorem 14 *Consider an orthogonal TVS-valued cone metric space $X = \mathbb{R}^n$ with the Euclidean metric $d(x,y) = \|x - y\|e^t$. The following hypothesis is satisfied if $\sum_{j=1}^{n} |\alpha_{ij}| \leq \alpha < 1$ for all $i = 1, 2, \ldots$*

(i) *there exists*

$$\|Ax - Ay\| = \|A\| \, \|x - y\|e^t,$$

where

$$\|A\|_F = \Big(\sum_{ij}^{n} |a_{ij}|^2\Big)^{\frac{1}{2}} = tr(A * A)^{\frac{1}{2}} = \sqrt{mn}M \leq \tau,$$

and M is modulus of the largest element of A.

(i) *for $x, y \in X$ such that $x \perp y$, we have*

$$d(x,y) = \|x - y\|e^t \preceq \psi(\tau \mathcal{M}_S(x,y)), \tag{34}$$

where

$$\mathcal{M}_S(x,y) = [d(x,y)]^{\gamma_1}, \Big[\max\{d(x,y), d(x, \Upsilon y), d(y, \Upsilon x)\} \Big]^{\gamma_2}. \Big[\frac{d(x, \Upsilon x).d(y, \Upsilon y)}{d(x,y)} \Big]^{1-\gamma_1-\gamma_2}. \tag{35}$$

Then, the system of linear equations (32) in n unknowns has a unique fixed point.

Proof To find a fixed point, we assume that $A = (a_{ij})_{mn}$ is an orthogonal matrix of order $m \times n$. Define a map $\Upsilon : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Upsilon x = Ax + b, x \in \mathbb{C}^n. \tag{36}$$

Using the Cauchy-Schwartz inequality, we have

$$\Big|\sum_{j=1}^{n} a_{ij}(x_j - y_j)\Big| \leq \Big(\sum_{j=1}^{n} \|a_{ij}\|^2\Big)^{\frac{1}{2}} \|x-y\|e^t.$$

Since Υ is an orthogonal mapping, and $x \perp y \in X$, using conditions (i) and (ii), we have

$$\begin{aligned}
\|\Upsilon x - \Upsilon y\|e^t &\leq \|(Ax-b)-(Ay-b)\| \\
&\leq \|Ax - Ay\| \\
&\leq \|A\|\,\|x-y\|, \\
&\leq \Big(\sum_{j=1}^{n} \|a_{ij}\|^2\Big)^{\frac{1}{2}} \|x-y\|e^t, \\
&\leq \sqrt{mn}M\|x-y\|e^t, \\
d(\Upsilon x, \Upsilon y) &\leq \tau d(x,y).
\end{aligned}$$

Consequently, we have

$$d(\Upsilon x, \Upsilon y) \preceq \psi(\tau \mathcal{M}_S(x,y)),$$

where

$$\begin{aligned}
\mathcal{M}_S(x,y) = [d(x,y)]^{\gamma_1}, \Big[\max\{d(x,y), d(x,\Upsilon y), d(y,\Upsilon x)\}\Big]^{\gamma_2}. \\
\Big[\frac{d(x,\Upsilon x).d(y,\Upsilon y)}{d(x,y)}\Big]^{1-\gamma_1-\gamma_2}.
\end{aligned}$$

This demonstrates that the conditions outlined in Theorems 3.2 and 14 are met. Therefore, the system of linear equations (32) with n unknowns possesses a unique fixed point. This concludes the proof. □

5 Conclusions

The application of fixed point theorems in interpolative orthogonal relational TVS-valued convex cone metric spaces has provided significant advancements in solving differential and matrix equations. By employing interpolative mappings and relation-theoretical approaches, a robust framework is established for identifying unique solutions and confirming the existence and uniqueness of fixed points. This methodology not only generalizes classical fixed point results but also introduces new tools and perspectives for addressing complex mathematical problems, ensuring consistent and reliable solutions for systems of equations. The findings contribute to a broader understanding of mathematical structures and their applications in various disciplines.

This paper extends previous work by developing fixed point theorems for interpolative relation-theoretical convex mappings in orthogonal TVS-valued cone metric spaces. Notable contributions include a fixed point theorem for orthogonal interpolative almost Istratescu contractions of type D and ψ-orthogonal interpolative Chatterjea-Reich-Rus-Ćirić-Kannan-type contraction mappings. These results generalize well-known theorems in the literature and are supported by illustrative examples. Additionally, the application of these results to the existence of solutions for differential and matrix equations demonstrates their practical significance. Theorems 9 and 10 were shown to satisfy the necessary hypotheses, establishing that systems of linear equations in n unknowns possess unique fixed points, thus completing the theoretical framework.

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Author contributions
L.W -prepared the whole manuscript.

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Code availability
There is no code required in this paper.

Declarations

Consent for publication

Competing interests
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