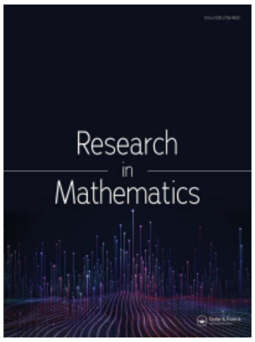

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SHORT COMMUNICATION

On a fixed point theorem of Karapinar for interpolative contractions

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ABSTRACT
We revisit the Karapinar's fixed point theorem for interpolative Kannan type contractions on a complete metric space, and establish its generalization.

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1. Introduction

The study of a class of maps $\mathcal{T} : (\mathsf{X}, \mathfrak{p}) \to (\mathsf{X}, \mathfrak{p})$ on a complete metric space satisfying certain condition(s) involving the metric $\mathfrak{p}$ was triggered in 1922 by Banach (1922) discovery of the celebrated Banach contraction principle (BCP), which assert that the map $\mathcal{T}$ satisfying

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \le \mu \mathfrak{p}(r, s) \quad \forall r, s \in \mathsf{X} \quad \text{for some} \quad 0 \le \mu < 1 \tag{1.1}$$

admits a *unique* fixed point. Clearly, (1.1) implies that $\mathcal{T}$ is a continuous map. This fact prompted researchers of that time to search for extensions/generalizations of BCP that do not require continuity of the map involved.

In 1968, Kannan (1969a) proved that the mentioned conclusion of BCP holds for the class of maps that we will refer to as *Kannan contractions*, which consists of the maps $\mathcal{T}$ satisfying

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \le \mu \left[\mathfrak{p}(r, \mathcal{T}r) + \mathfrak{p}(s, \mathcal{T}s)\right] \quad \forall r, s \in \mathsf{X} \quad \text{for some} \quad 0 \le \mu < \frac{1}{2}. \tag{1.2}$$

He further proved in Kannan (1969b) that the conditions (1.1) and (1.2) are independent, and the latter does not imply that $\mathcal{T}$ is continuous. Remarkably, it was established by Subrahmanyam (1974) that a metric space $(\mathsf{X}, \mathfrak{p})$ is complete if and only if every $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ satisfying (1.2) has a fixed point. So, the above Kannan fixed point theorem characterizes completeness of $(\mathsf{X}, \mathfrak{p})$, which is not the case for BCP as observed by Connell (1959).

In 2018, Karapınar (2018) introduced another interesting class of *interpolative Kannan type contractions* consisting of the maps $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ such that

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \le \mu \left[\mathfrak{p}(r, \mathcal{T}r)\right]^{\beta} \left[\mathfrak{p}(s, \mathcal{T}s)\right]^{1-\beta} \quad \text{for some} \quad 0 \le \mu < 1, \text{ and } 0 < \beta < 1. \tag{1.3}$$

Remark 1.1. *Denoting the set of fixed points of* $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ *by* FS, *i.e. FS* $:= \{p \in \mathsf{X} : \mathcal{T}p = p\}$, *we observe that the condition (1.3) does not hold for distinct points* r, s *if* r *or* s *is contained in* FS *since* $\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \ge 0$. *So, (1.3) is required to hold for all* $r, s \in \mathsf{X} - FS$.

As described in Karapinar et al. (2018), he introduced (1.3) by mimicking the form of an inequality appearing in interpolation theory for Banach spaces. Moreover, unlike BCP, an interpolative Kannan type contraction defined on a complete metric space admits *at least* one fixed point. This result has been stimulating a lot of research works in fixed point theory and its applications (see for example (Abbas et al., 2023; Debnath & de La Sen, 2019; Devi & Debnath, 2025; Nazam et al., 2022, Hussain, 2024; Nazam et al., 2023; Nazam et al., 2021; Javed et al., 2022; Konwar & Debnath, 2022; Konwar et al., 2022; Tomar et al., 2024; Wangwe et al., 2024)).

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In this article, we revisit the aforementioned result due to Karapinar and establish its generalization.

2. Main results

We now present our results.

Definition 2.1. Let $(\mathsf{X}, \mathfrak{p})$ be a metric space. A map $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ is called a *multiplicative Kannan type contraction* (MKC) if there exit real numbers β and μ with $0 \leq \mu < 1$ such that

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leq \mu\,[\mathfrak{p}(r, \mathcal{T}r)]^{\beta} [\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta} \quad \text{for all} \quad r, s \in \mathsf{X} - FS. \tag{2.1}$$

Remark 2.2. *Constraining β by $0<\beta<1$, MKC reduces to an interpolative Kannan contraction.*

Theorem 2.3 *Let $(\mathsf{X}, \mathfrak{p})$ be a complete metric space. Any multiplicative Kannan type contraction $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ has a fixed point.*

Proof**.** We proceed in two steps:

1. Fix any $s_0 \in \mathsf{X}$, generate the sequence $\{s_n\} \subset \mathsf{X}$ by $s_n = \mathcal{T}s_{n-1}$ for any positive integer n, and establish that $\{s_n\}$ is a Cauchy sequence;
2. Show that the limit point of the sequence $\{s_n\}$,which exists since X is complete, is a fixed point of $\mathcal{T}$.

Observe that if $\mathcal{T}s_0 = s_0$, the conclusion of the theorem holds. And otherwise, we can assume without loss of generality that $s_n \neq s_{n+1}$, and so $\mathfrak{p}(s_n, s_{n+1})>0$.

Since $\mathcal{T}$ is MKC, then using (2.1) we get

$$\mathfrak{p}(s_{n+1}, s_n) \leq \mu\,[\mathfrak{p}(s_n, s_{n+1})]^{\beta} [\mathfrak{p}(s_{n-1}, s_n)]^{1-\beta} \quad \text{for any real number} \quad \beta. \tag{2.2}$$

The rest of the proof involves simplifications and use of the triangle inequality. We have

$$\begin{aligned} [\mathfrak{p}(s_n, s_{n+1})]^{1-\beta} &\leq \mu\,[\mathfrak{p}(s_{n-1}, s_n)]^{1-\beta}, \\ \mathfrak{p}(s_n, s_{n+1}) &\leq \mu\mathfrak{p}(s_{n-1}, s_n), \\ &\leq \mu^n \mathfrak{p}(s_1, s_0). \end{aligned} \tag{2.3}$$

Consequently, for positive integers m and n with $n < m$ we have

$$\begin{aligned} \mathfrak{p}(s_n, s_m) &\leq \mathfrak{p}(s_n, s_{n+1}) + \ldots + \mathfrak{p}(s_{n+m-1}, s_{n+m}), \\ &\leq (\mu^n + \ldots + \mu^{n+m-1})\,\mathfrak{p}(s_1, s_0), \\ &\leq \frac{\mu^{n+m}}{1-\mu}\mathfrak{p}(s_1, s_0) \quad \text{since } \mu \in [0, 1). \end{aligned} \tag{2.4}$$

So, $\{s_n\}$ is a Cauchy sequence, since $\mathfrak{p}(s_n, s_m)$ approaches 0 as n and m tend to ∞. This establishes step (1).

Since X is complete, then $\{s_n\}$ converges to some $s \in \mathsf{X}$. We conclude the proof by showing that s must be a fixed point of $\mathcal{T}$. In deed, we have

$$\begin{aligned} \mathfrak{p}(s, \mathcal{T}s) \quad &\leq \mathfrak{p}(s, \mathcal{T}s_n) + \mathfrak{p}(\mathcal{T}s_n, \mathcal{T}s), \\ &\leq \mathfrak{p}(\mathcal{T}s_n, \mathcal{T}s) \leq \mu[\mathfrak{p}(s_n, \mathcal{T}s_n)]^{\beta}[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}\text{using(2.1)}, \\ &\leq \mu\mathfrak{p}(s, \mathcal{T}s)\text{as}n \to \infty. \end{aligned} \tag{2.5}$$

Thus, $\mathfrak{p}(s, \mathcal{T}s) = 0$, and therefore $\mathcal{T}s = s$.

The following examples demonstrate the novelty of our results above.

Example 1. *Consider the complete metric space $(\mathsf{X}, \mathfrak{p})$, consisting of the set $\mathsf{X} = \{0, 2, 4, 6, 8\}$ and the metric $\mathfrak{p}(r, s) = |r - s|$. Define the function $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ by*

$$\mathcal{T}r = \begin{cases} r & \textit{if} \quad r \in \{0, 2\}, \\ 4 & \textit{otherwise}. \end{cases} \tag{2.6}$$

Clearly, $\mathcal{T}$ satisfies (2.1) for any β and μ since

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) = 0 \quad \textit{for} \quad (r, s) \in \{(4, 4), (4, 6), (4, 8), (6, 4), (6, 6), (6, 8), (8, 4), (8, 6), (8, 8)\}. \tag{2.7}$$

So by Theorem 2.3, $\mathcal{T}$ has a fixed point. Moreover, there are a couple of things to note:

1. *MKC may admit multiple fixed points since $FS = \{0, 2\}$;*
2. *$\mathcal{T}$ is not an interpolative Kannan type contraction for $\beta \notin (0, 1)$;*
3. *$\mathcal{T}$ is not a Kannan contraction since*

$$\begin{aligned} 2 = \mathfrak{p}(\mathcal{T}6, \mathcal{T}2) &\not\leq \mu[\mathfrak{p}(6, \mathcal{T}6) + \mathfrak{p}(2, \mathcal{T}2)] \\ = 2\mu \quad &\textit{for any} \quad 0 \leq \mu < \frac{1}{2}. \end{aligned} \tag{2.8}$$

Example 2. *Consider the self map $\mathcal{T}r = \begin{cases} \frac{1}{2} & \textit{if } r = 0, \\ 1 & \textit{otherwise} \end{cases}$ on a complete metric space $([0, 1], \mathfrak{p})$, where $\mathfrak{p}(r, s) = |r - s|$. We next show that $\mathcal{T}$ is MKC as follows.*

- *Case 1: when $r = s = 0$*
  We have $\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) = 0$, so $\mathcal{T}$ satisfies (2.1) for any $0 \leq \mu < 1$ and β.
- *Case 2: when $r, s \in (0, 1)$*
  Similar to Case 1 above.

- *Case 3: when* $\mathrm{r}=0$ *and* $s \in (0,1)$
  Imposing (2.1) we see that

$$\frac{1}{2} \leq \mu \left[\frac{1}{2}\right]^{\beta} [1-s]^{1-\beta} < \mu \left[\frac{1}{2}\right]^{\beta}$$

  holds for $\beta = -1$ *and* $\mu = \frac{1}{4}$. *(Note that* $\mathcal{T}$ *is not an interpolative Kannan type contraction.)*
- *Case 4: when* $r \in (0,1)$ *and* $\mathrm{s}=0$
  Similar to Case 3 above.

Thus, $\mathcal{T}$ *satisfies (2.1) for* $\beta = -1$ *and* $\mu = \frac{1}{4}$. *So, by Theorem 2.3, we conclude that* $\mathcal{T}$ *admits a fixed point.*

In view of Remark 2.2, we deduce Theorem 2 in Karapınar et al. (2018), which is the corrected version of Theorem 2.2 in Karapınar (2018), stating that:

Corollary 2.4. *Any interpolative Kannan type contraction on a complete metric space has a fixed point.*

Next, we modify our Definition 2.1 and define the following contractions of rational type, and obtain a fixed point theorem (Corollary 2.6) for the same.

Definition 2.5. Let $(\mathsf{X}, \mathfrak{p})$ be a metric space. A map $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ is called a *sub-multiplicative Kannan type contraction* (sMKC) if there exist real numbers μ, β and γ with $0 \leq \mu < 1$ and $\gamma \geq 1$ such that

$$\mathfrak{p}(\mathcal{T}r, \mathcal{T}s) \leq \frac{\mu\,[\mathfrak{p}(r, \mathcal{T}r)]^{\beta}\,[\mathfrak{p}(s, \mathcal{T}s)]^{1-\beta}}{\mathcal{F}(r,s)} \quad \text{for all} \quad r, s \in \mathsf{X} - FS, \quad \text{where} \tag{2.9}$$

$$\mathcal{F}(r,s) := \gamma + \max\{\mathfrak{p}(r,s), \mathfrak{p}(r,\mathcal{T}r), \mathfrak{p}(s,\mathcal{T}s), \mathfrak{p}(\mathcal{T}r,s), \mathfrak{p}(r,\mathcal{T}s), \mathfrak{p}(\mathcal{T}r,\mathcal{T}s)\}. \tag{2.10}$$

Corollary 2.6. *Let* $(\mathsf{X}, \mathfrak{p})$ *be a complete metric space. Any* sub-multiplicative *Kannan type contraction* $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ *has a fixed point.*

Proof**.** The proof is analogous to that of Theorem 2.3, and we proceed as in that proof. Since $\mathcal{T}$ is sMKC, then it satisfies (2.9)

$$\begin{aligned}\mathfrak{p}(s_{n+1}, s_n) &= \mathfrak{p}(\mathcal{T}s_n, \mathcal{T}s_{n-1}) \\ &\leq \frac{\mu\,[\mathfrak{p}(s_n, s_{n+1})]^{\beta}\,[\mathfrak{p}(s_{n-1}, s_n)]^{1-\beta}}{\mathcal{F}(s_n, s_{n-1})}.\end{aligned}$$

Simplifications then give

$$\begin{aligned}[\mathfrak{p}(s_n, s_{n+1})]^{1-\beta} &\leq \frac{\mu\,[\mathfrak{p}(s_{n-1}, s_n)]^{1-\beta}}{\mathcal{F}(s_n, s_{n-1})}, \\ &\leq \mu\,[\mathfrak{p}(s_{n-1}, s_n)]^{1-\beta} \quad \text{since} \quad \mathcal{F}(r,s) \geq 1,\end{aligned} \tag{2.11}$$

Thus,

$$\mathfrak{p}(s_n, s_{n+1}) \leq \mu \mathfrak{p}(s_{n-1}, s_n).$$

Using the triangle inequality, we then get

$$\mathfrak{p}(s_n, s_{n+1}) \leq \mu^n \mathfrak{p}(s_1, s_0).$$

The rest of the proof is the same as that of Theorem 2.3.

Remark 2.7. *Note that the map* $\mathcal{T}$ *in Example 1 satisfies (2.9) for any* γ. *So, using our Corollary 2.6, we can also conclude that* $\mathcal{T}$ *has a fixed point. However, Corollary 2.6 does not apply to Example 2 since the map* $\mathcal{T}$ *does not satisfy (2.9).*

3. Conclusions

We have introduced the notions of *multiplicative Kannan type contraction* and *sub-multiplicative Kannan type contraction*, and establish fixed point theorems for the classes of such contractions. Our main results generalize/extend fundamental fixed point results for *interpolative Kannan type contractions* due to Karapınar (2018); Karapınar et al., (2018).

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