INVESTIGATION OF SOME FIXED POINT THEOREMS FOR VARIOUS TYPES OF MAPPINGS IN ABSTRACT SPACES

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By

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A Thesis Submitted in Fulfilment of the Requirements for the Degree of Doctor of Philosophy (Mathematics) of the University of Dar es Salaam

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CERTIFICATION

The undersigned certify that they have read and hereby recommend for acceptance by the University of Dar es Salaam a thesis entitled: ***Investigation of Some Fixed Point Theorems for Various Types of Mappings in Abstract Spaces***, in fulfilment of the requirements for the degree of Doctor of Philosophy (Mathematics) of the University of Dar es Salaam.

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I, **Lucas Wangwe**, declare that this thesis is my own original work and that it has not been presented and will not be presented to any other University for a similar or any other degree award.

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DEDICATION

To my parents, who have been a constant source of support and encouragement during the challenges of my life. My dearest wife Mary Lucas and my children, whom I am truly thankful for having you in my life. My beloved brothers, sisters and friends.

ABSTRACT

Fixed point theory is a fundamental tool in nonlinear functional analysis. It has many applications e.g. in Approximation theory, Optimisation theory, Variation inequalities, Game theory and Economics etc. The fixed point theory is a powerful tool to determine existence and uniqueness of the solutions of Differential equations, Integral equations, Partial differential equations, Fractional differential equations, Matrix equations and Functional equations. A fixed point problem can be stated as follows: Let X be a non-empty set and $T : X \to X$ be a mapping. A point $x \in X$ is a fixed point or invariant point of the mapping T if $Tx = x$. Does a fixed point exist for every map? Moreover, if such a point exists, is it unique, and how can we find it? We can distinguish three major approaches in fixed point theory: metric approach, topological approach, and discrete approach. Historically, these approaches were initiated by the discovery of major theorems: Brouwer fixed point theorem, Banach fixed point theorem, and Tarski fixed point theorem. In this thesis, we are concerned with the second approach, the metric fixed point theory. Fixed point theory and Banach contraction principle have been studied and generalised in different spaces, and various fixed-point theories were developed. Hence, this study investigated the fixed point theorems for various types of constructive mappings in various abstract spaces. The Banach contraction method has been used to obtain the fixed point theorems and their applications to ordinary and fractional differential equations. This study showed several ways to construct, extend, formulate, prove, and generalise fixed point theorems in abstract spaces using various maps, i.e., single valued maps, multivalued maps, hybrid maps and implicit maps. Also, the generalisation is done by considering relatively large classes of abstract spaces; Cone metric space, b-metric space, partial b-metric spaces, metric-like spaces, partial metric spaces, quasi partial S_b-metric-like spaces, and G-metric spaces. Finally, the proofs of the results are established by finding coincidence points or common fixed points.

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LIST OF ABBREVIATIONS AND SYMBOLS

E. A.	EL-Mountawakil and Aamri property.
CLR	Common limit range property.
JCLR	Joint common limit range property.
HDE	Hybrid differential equation.
$\in, \notin$	belongs to and does not belong to.
$\emptyset$	empty set.
$\forall n$	for all n.
et al.	et alii (and others).
$\mathbb{R}^n$	n-dimension Euclidean space.
C(S,T)	the set of coincidence point of S and T.
max or min	maximum or minimum.
τ	tau.
inf	infimum (or greatest lower bound).
sup	supremum(or least upper bound).
$\mathbb{N}$	set of natural numbers.
$\mathbb{N}_0$	set of whole numbers.
$\mathscr{F}$	The family of all functions $F \in \mathscr{F}$.
σ	greek letter sigma.
η	greek letter eta.
ζ	greek letter zeta.
Υ	Upsilon.
ϑ	vartheta.
ρ	rho.
ϕ	phi.
φ	varphi.
ψ	psi.

CHAPTER ONE

INTRODUCTION

The study of fixed points has attracted the interests of a number of researchers from different areas over the past century. Fixed point theory is a fascinating area of research for the researches studying non linear phenomena. It has many applications to the nonlinear functional analysis, Approximation theory, Optimization Theory (Saddle function), Variation inequalities, Game theory (Nash equilibrium) and Economics (Black Scholes theorem). Fixed point theory is quite and sequel in the existing theory of Differential, Integral, Partial, Fractional differential and functional equations. Fixed point theory as well as Banach contraction principle have been studied and generalised in different spaces and various fixed point theorems are developed.

1.1 General Introduction

Nonlinear analysis is a significant branch of mathematics wherein fixed point theory lies in its heart. Indeed, fixed point theory offers an elementary, vigorous and effective tool for nonlinear analysis. Moreover, it has fruitful applications in mathematics and many other various science disciplines. Consequently, this theory has gained a remarkable scope of research and attracted many researchers leading the development of this theory in several directions. The strength of the fixed point theory lies in its wide variety of applications. Indeed, fixed point theory is a powerful tool to determine the existence and uniqueness theories of functional equations, integral equations, matrix equations, ordinary as well as partial differential equations, random differential equations, variational inequalities, etc., besides facilitating various problems arising in different domains, such as approximation theory, differential geometry, eigenvalue problems, functional analysis, operator theory and topology.

The fixed point theory deals with various mathematical models representing phenomena arising in chemical equations, control theory, electrical heating in Joule-Thomson effect, equilibrium points in economics, fluid flow, fractal theory, Nash equilibrium in game theory, neutron transport theory, optimisation theory, potential theory, probability theory, steady-state temperature distribution, etc. Moreover, many practical and research problems in various fields beyond mathematics can reduce to fixed point problems, which include biology, chemistry, computer science, economics, engineering, global analysis statistics, operations research, physics, etc. Though the existence or non-existence of a fixed point is an intrinsic property of a mapping, there do exist many necessary or sufficient conditions for the existence of fixed points involving a mixture of topological, order-theoretic or geometric properties on the mapping or its domain.

Several mathematical problems arising in different areas of non-linear analysis, such as Functional analysis, Approximation theory, Optimisation theory, Variation inequalities, Game theory and Economics, can be modelled by the equation

$$x \quad = \quad T^i x, \tag{1.1}$$

where $i = 0, 1, 2, \ldots,$ T is generally a non linear operator and x, an element of metric space X. The solutions to this equation are fixed points of T and the theorems concerning the existence and properties of fixed points are known as fixed point theorems.

Such theorems have broad applications in proving the existence and uniqueness of the solutions of nonlinear functions such as Differential, Integral, Partial and Functional Equations and in several branches of analysis, topology, biological science, chemistry, physics and many other applied sciences in various abstract spaces (Conrad 2009).

A fixed point of a self mapping $T : X \longrightarrow X$ is a point $x \in X$ such that $Tx = x$. A mapping can have one or more than one fixed point, for example, a function $T : [0,1] \longrightarrow [0,1]$ defined by $Tx \ = \ x^2$ has two fixed points. That is, $x = 0$ and $x = 1$ are fixed points of T since $T0 = 0$ and $T1 = 1$. Also, not every map needs to have a fixed point. For example, a translation map does not have a fixed point. In this study, we are interested in the map having one and a unique fixed point.

Historically, the roots of fixed point theory referred to mathematical activities of great mathematicians, namely: Liouville (1834), Cauchy (1840), Poincare (1886), Lipschitz (1877), Peano (1890), Picard (1890) and some others. The initiation of fixed theory can be traced back formally to the beginning of the twentieth century in the pioneering article of the Dutch mathematician Brouwer (1912).

The study of fixed point theorems is mainly divided into three main areas (Chetan 2017, Agarwal *et al.* 2018):

(i) Topological fixed point theorems $\Longrightarrow$ Non-constructive,

(ii) Metric fixed point theorems $\Longrightarrow$ Constructive,

(iii) Discrete fixed point theorems $\Longrightarrow$ Non-constructive.

These approaches were initiated by discovering three significant theorems: Brouwer fixed point theorem, Banach fixed point theorem, and Tarski's fixed point theorem.

Brouwer (1912) stated a fixed point theorem for topological spaces, which is as follows:
Theorem 1.1 *(Brouwer 1912) Let* B *be a closed ball in* $\mathbb{R}^n$*. Then any continuous mapping* $T : B \longrightarrow B$ *has at least one fixed point.*

Brouwer's fixed point theorem asserts the existence of a fixed point whenever B is the unit ball in $\mathbb{R}^n$ and T is continuous. Where the spaces are subsets of $\mathbb{R}^n$, are not of much use in nonlinear functional analysis where one is generally concerned with infinite-dimensional subsets of some function spaces. However, several researchers gave extensions and proof of this theorem of topological nature. The first extension was investigated by Birkhoff and Kellogg (1922). Later, Schauder (1930) extended Brouwer's theorem to the case where X is a compact and convex subset of a normed linear space, where the fixed point lies in the space of functions, and this point may be a function that solves a nonlinear integral equation or partial differential equation. This theorem was extended to locally convex topological vector spaces by (Tychonoff 1935). Note that Birkhoff-Kellogg (1922), Schauder (1930), and Tychonoff (1935) applied their results in the existence of solutions of differential and integral equations. Further, these results were extended by Kakutani (1941). Brouwer's theorem applies to continuous point-to-point functions. Kakutani dealt with set-valued functions; i.e., point-to-set functions. However, the theorem can not tell about the uniqueness and determination of the solution.

Tarski (1955) formulated and proved an elementary fixed point theorem in arbitrary complete lattices. It is one of the essential results in lattice theoretic fixed point theory because of its wide range of applications in theories of ordered sets, real functions, Boolean algebra, as well as general set theory and topology. However, the proofs in all articles, as mentioned earlier, were non-constructive.

In this study, we are interested in the second approach, the metric fixed point theory. Metric fixed point theory is a sub-branch of fixed point theory containing methods and results that involve properties of an essentially isometric nature. The first result of this setting was introduced by the Polish mathematician Banach (1922). His result oftenly referred as the "Banach contraction principle" (BCP). This principle ensures a unique fixed point for every contraction mapping defined on a complete metric space. Though the Banach contraction principle is straightforward and natural, the involved condition of contraction is very restrictive. Nevertheless, it is one of the most important and valuable results in the metric fixed point theory with constructive proof.

There are several methods in functional analysis which are applied in solving different mathematical problems; one of them, which is very popular and most used, is the fixed point method. The method uses the Banach contraction principle and the Picard iteration procedure to obtain solutions.

The contractions mapping are of fundamental importance for analysis and essential tools for proving the existence of the solution of ODE. To present the Banach contraction principle, we need to introduce the concept of Lipschitz mappings.

Definition 1.1 *(Banach 1922) Let* (X,d) *be a metric space. A mapping* $T : X \to X$ *is said to be Lipschitzian if there exist a constant* $\alpha \geq 0$ *(called Lipschitz constant) such that*

$$d(Tx,Ty) \leq \alpha d(x,y), \forall\, x,y \in X. \tag{1.2}$$

A condition of type (1.2) *is called Lipschitz condition, where* $\alpha \geq 0$ *and* α *is called*

Lipschitz constant. Contractions are Lipschitz maps with a Lipschitz constant less than 1.

Consider $X = [0,1]$. A mapping $T : X \to X$ with the usual metric d. Defined as $d(x,y) = |x-y|$, $\forall\, x,y \in X$. To see this, consider the equation $x^3+x^2-7x+2=0$ for $0 \leq x \leq 1$, it is easy to show that the equation is a contraction and has only one fixed point. The equation can be written in the form of fixed point, that is $x = Tx$

$$x = \frac{1}{7}(x^3+x^2+2). \tag{1.3}$$

Define a mapping T by $Tx = \frac{1}{7}(x^3+x^2+2)$, we have

$$Tx = \frac{1}{7}(x^3+x^2+2).$$

Using (1.3), we get

$$\begin{aligned} d(Tx,Ty) &\leq \left|\frac{1}{7}(x^3+x^2+2)-\frac{1}{7}(y^3+y^2+2)\right|, \\ &\leq \frac{1}{7}\left|x^3-y^3+x^2-y^2\right|, \\ &\leq \frac{1}{7}\left|(x-y)(x^2+xy+y^2+x+y)\right|, \\ &\leq \frac{1}{7}\left|x-y\right|\left|x^2+xy+y^2+x+y\right|, \\ &\leq \frac{1}{7}d(x,y)\left|1^2+1\times 1+1^2+1+1\right|, \\ &\leq \frac{1}{7}d(x,y)|5|, \\ &\leq \frac{5}{7}d(x,y), \end{aligned}$$

which shows that Tx is a contraction mapping with a Lipschitz constant $\frac{5}{7}$. Since $\frac{5}{7} < 1$. Therefore, T has a fixed point.

Banach (1922) established a fixed point theorem for single-valued contractive mappings in metric spaces, which is known as Banach contraction principle (*BCP*), stated as follows:

Definition 1.2 *(Banach 1922) Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a contraction if there is $k \in [0,1)$ such that*

$$d(Tx,Ty) \leq kd(x,y), \forall x,y \in X. \tag{1.4}$$

Theorem 1.2 *(Banach 1922) Let (X,d) be a complete metric space and $T : X \to X$ a contraction mapping. Then T has a unique fixed point or $Tx = x$ has a unique solution.*

Furthermore, the Banach contraction principle guarantees that the Picard sequence of T based at any point converges to the fixed point, i.e., starting at any point $x_0 \in X$, the repeated iterations of the mapping at x_0 yields a sequence that converges to the unique fixed point of T. The advantage of this principle is that its hypothesis is very simple and always gives a unique fixed point that can be determined using a constructive method. The only disadvantage attached to this principle is that assuming the mapping to be contraction forces the mapping T to be continuous at each point of the space. However, this principle is widely considered as the source of metric-theoretical fixed point theory and one of the most fundamental and powerful tools of nonlinear analysis.

In general, the Banach fixed point theorem and its generalisation does not only claim the existence and uniqueness but also yields an algorithm for finding it. There are five basic steps to follow in proving the results:

(i) Construct picard iteration of a described sequence.

$$u_n = T^n u_0.$$

(ii) Apply the contraction mapping (Given/extended).

(iii) Convert the iterated sequence $\{u_n\}$ to be a Cauchy sequence.

(iv) Since X is complete, $\{u_n\}$ converges to say u^*. Now we have to prove that u^* is a fixed point of the operator T, i.e., $Tu^* = u^*$.

(v) Proving the uniqueness: To prove the uniqueness, we assume that v^* is another fixed point of the operator T, i.e., $Tv^* = v^*$. Apply to the given contraction to prove that $u^* = v^*$. Hence u^* is unique fixed point of the operator T.

The Banach contraction principle (BCP) is a constructive structure. That is,

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{T} \begin{pmatrix} Tx \\ Ty \end{pmatrix} \Longrightarrow \begin{pmatrix} d(x,y), d(x,Tx), d(y,Ty), d(x,Ty) \\ d(y,Tx), d(Tx,Ty) \end{pmatrix}.$$

The custom of improving contraction conditions in fixed point theory is still in demand. The motivation of this study is to prove, extend, improve and generalise various types of constructive mappings in abstract spaces with some applications to fractional differential equations and ordinary differential equations by using the Banach contraction principle method.

1.2 Statement of the Problem

Metric fixed point theory comprises such fixed point results wherein geometric properties on the abstract space and underlying mapping are effectively utilised. This type of research is relatively not new, but still a new area of study. Although a substantial number of definitive results have already been discovered, many others are still open. In addition, there are many questions awaiting answers regarding the limits to which the theory may extend. Some of these questions are merely tantalising, while others suggest substantial new avenues of research.

In answering some of these questions, Poincare (1886) was the first to work in this field and proved the fixed point theorem for a function $f(x) = x$. Fréchet (1906) introduced the study of metric fixed point theory in abstract spaces. Then, Brouwer (1912), proved fixed point theorem for the function $f(x) = x$ in $\mathbb{R}^3$. He also proved the fixed point theorem for a square, a sphere and their n-dimensional counterparts, further extended by Kakutani (1941). Brouwer's theorem applies to continuous point-to-point functions. Kakutani deals with set-valued function; i.e., point-to-set functions. However, the theorem can not tell about the uniqueness and determination of the solution.

Banach (1922) established the contraction mapping principle in metric space, where he proved the existence and uniqueness of a fixed point $(f(x) = x)$. The fixed point theory, as well as the Banach contraction principle, has been studied and generalised by several authors in different abstract spaces, namely; Kannan (1968), Nadler (1969), Jungck (1976), Sessa (1982), Khan 1984, Czerwik (1993), Matthew (1994), Popa (1997), Aamri and Moutawakil (2002), Nieto and Rodrigurz-Lopez (2005), Kaewcharoen and Kaewkhao (2011), Situnavarnt and Kumam (2011), Aydi *et al.* (2012), Wardowski (2012), Ansari (2014) and many others.

The importance of these generalisations shows that the quantity and diversity of such metric fixed point theory are constantly increasing due to the discovery of a new system of equations in different dynamic systems. Motivated by all of the above types of generalisations of Banach contraction principle, This study intends to investigate some fixed (coincidence) point theorems for various types of constructive mappings in abstract spaces.

Specifically, we improve some of the following works: Morandi and Alimohammadi (2011), Imdad *et al.* (2014), Ahmadullah *et al.* (2016), Ali and Kamran (2016), Gopal *et al.* (2017), Qawaqneh *et al.* (2019), Eke *et al.* (2019), Batra *et al.* (2020), Rao *et al.* (2020), Aserkar and Gandhi (2020), Shoib *et al.* (2020), Kanwal *et al.* (2020), Karapinar *et al.* (2020), Gauntam and Verma (2021).

The study will generalise and improve some of the existing fixed point theorems in the literature using the Banach contraction principle concept. Also, we give some applications to nonlinear fractional differential equations, mixed Volterra-Fredholm integral equations and ordinary differential equations.

1.3 Research Objectives

This research aimed to construct, extend, prove and generalise fixed point and common fixed point theorems using various types of contractive mappings with some applications in abstract spaces.

The specific objectives of this study were as follows:

(i) To construct and prove common fixed point theorems for F-Kannan contraction mapping and F-Kannan Suzuki mappings in generalised metric spaces with some applications to fractional differential equations.

(ii) To extend and prove fixed point theorems for multi-valued contractive mappings, in partial metric space and b-metric space with some applications to Volterra integral equations and ordinary differential equations.

(iii) To prove fixed point theorems for hybrid pairs for non-self mappings in weak partial b-metric space and p-hybrid mappings using common limit range property in G-metric space with some applications to hybrid differential equations.

(iv) To formulate and prove common fixed point theorems for implicit mappings in metric-like space and quasi-S_b metric space with some applications to the integral-differential equation and second-order differential equations.

1.4 Research Questions

The following research questions were used to guide this study:

(i) What are sufficient conditions to construct and prove a fixed point theorem for a single-valued map and give new results by extending the structure of the space?

(ii) What are the properties of the new extended fixed point theorems for multi-valued maps on abstract spaces equivalent to previously known spaces?

(iii) How can a fixed point theorem for hybrid maps on a new generalised metric space be directly obtained from a fixed point theorem or a known metric space?

(iv) How can we formulate and prove fixed point theorems for implicit maps by weakening the contraction condition of the map and giving a sufficiently rich structure of the space to compensate for the relaxation of the contraction condition be valuable?

1.5 Significance of the Study

Researchers have generated many fixed point theorems for various types of contractive mappings in abstract spaces. In this work we have generated various theorems for applying fixed point theorems in differential equations, integral equations, and fractional differential equations.

There are some fixed point theorems for non-self mappings using (JCLR) property much broader than that of self mappings theorems. The developed theorems for non-self mappings also apply to self mappings. However, there is a scope for research on fixed point theorems dealing with non-self mappings in abstract spaces.

This research aims to address some of these issues. In doing so, this study contributes to the knowledge and understanding of fixed point theory in abstract spaces. The findings of this study are expected to fill the gaps of the existing knowledge and benefit society at large; specifically, the paper produced will motivate more research to work on this field of study. Due to its wide applications in nonlinear analysis such as control theory, convex optimisation, differential inclusion and economics, this research will be fruitful for researchers in the near future.

1.6 Organisation of the Thesis

This thesis discuss the generalisations of Banach contraction principle, investigate and prove some fixed (coincidence) point theorems for various types of constructive mappings in the setting of abstract spaces and also explore the possibility of their applications. Each chapter of the thesis is divided into various sections and consistently the first section of each chapter provides an introduction to the content of chapter.

Chapter 1, comprises the general introduction, statement of the problem, research objectives and questions, significance of the study and lastly is the organization of the thesis. In Chapter 2, we give a detailed literature review which is related to this work. In Chapter 3, comprises proof of the coincidence fixed point theorems for a pairs of F-Kannan and F-Kannan-Suzuki mappings in generalised metric spaces.

Chapter 4, covers the multivalued fixed point results in ordered partially metric spaces and ordered partially b-metric spaces. In Chapter 5, we prove a common fixed point theorem for two hybrid pairs of non-self mappings in weak partial b-metric spaces and p-hybrid mappings in G-metric spaces with an applications. Chapter 6, incorporates the common fixed point theorem for a pair of self-mappings in metric-like spaces and quasi partial S_b-metric spaces for implicit contractive mappings related to binary relation. Chapter 7, contains the conclusions of our results, limitation of the study, the list of publications and some areas for future research are mentioned.

Definitions, theorems, corollaries and remarks are numbered per chapter and sequentially per section. In the end, the references are given, the list is by no means exhaustive but only books and papers referred to in the thesis are included. Notice that the references are arranged alphabetically according to the last names of the authors. At the end contains an appendix which shows the list of publications.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction and Preliminaries

This chapter presents the basic concepts and some previously known results. Some fundamental concepts and useful properties of various spaces are also given. Further, in the following chapters, we will prove extended versions of some of these results for various abstract spaces. Section 2.2 discusses the notion of single-valued contractive mapping in metric space with some previously known results in the literature. This concept will be used in chapter 3 to construct and establish fixed point theorem for F-Kannan mappings in generalised metric space. In section 2.3, we will give some basic concepts for various types of abstract spaces based in this thesis to develop the main results, specifically in chapter 3, chapter 4, chapter 5 and chapter 6. Section 2.4 discusses some multivalued notions for various types of abstract spaces using the Hausdorff metric, used in chapter 4 to prove fixed point theorems for multivalued mapping in partial metric and b-metric space.

Section 2.5 explains hybrid contractive mapping, which is essential, especially in chapter 5, to prove and generalise the results for hybrid pair for non-self mappings in weak partial b-metric space. Next, in section 2.6, we explain implicit function with it's related concepts. Finally, section 2.7 gives insights on order theoretic notion in a binary relation, used in chapter 6 to formulate and prove a common fixed point theorem for implicit mappings via $(E.A)$-property on metric-like space employed a binary relation.

Thus far, many results enriching the Banach contraction principle have been proved, and such interest is still on. One of the generalisations of this principle is obtained by considering a relatively more general contraction condition. For this kind of work, the reader is referred to Kannan (1968), Reich (1971), Chatterjea (1972), Hardy and Rogers (1973) and Ćiri ć (1974), Rhoades (1977), Ćiri ć (1993) and Ćiri ć (2005)). Further, Boyd and Wong (1969), Geraghty (1973), Matkowski (1977), Mukherjea (1977), Khan *et al.* (1984), Jotic (1995), Dutta and Choudhury (2008), Ansari (2014) and Wang and Chen (2019) obtained more general contraction conditions using different types of control functions.

Improving Banach contraction principle received a new impetus when the researchers attempted to prove generalised fixed point results. With such a quest, Popa (1997) initiated the idea of implicit function, another concept for generalising the Banach contraction principle. The concept has been used by Imdad *et al.* (2002), Berinde and Vetro (2012) and Amadullah *et al.* (2016) and several others.

On the other hand, another way of generalising the Banach contraction principle is to prove coincidence or common fixed point results, which is done by increasing the number of involved mappings, such as single-valued map (Banach 1922), multi-valued map (Nadler 1969), implicit map (Popa 1997) and hybrid map (Naimpally *et al.* 1986). The first result concerning coincidence point was by Machuca (1967), which was further improved by Goebel (1968), while Jungck (1976) proved the first-ever common fixed point theorem using two compatible mappings in metric space. Later, Sessa (1982) initiated the notion of commutativity mappings.

Furthermore, a way of generalising this principle is obtained by considering relatively larger classes of abstract spaces. Fréchet (1906) initiated the concept of Metric space, which was used as a bridge in the generalisation of abstract spaces. Matthews (1994) introduced non-zero self-distance, which is widely applied in computer networking, data structure, and Computer programming languages. Czerwik (1993) introduced the measure of the metric using b-metric space. Branciari (2000) introduced a class of generalised metric spaces by replacing triangular inequality with similar ones which involve four or more points instead of three and improved the Banach contraction mapping principle.

Thereafter, Mustafa and Sims (2006) gave a generalisation of D-metric space to G-metric space. They defined several notions as continuity, completeness, compactness, convergence, and space product in the G-metric space setting. Sedghi *et al.* (2012) gave a fascinating generalisation of D-metric space and G-metric space to S-metric space. Amini-Harandi (2012) gave a generalisation of partial metric spaces to metric-like spaces. Shukla (2014) provided an extension by combining partial metric space and b-metric space to partial b-metric space. Recently, Gupta and Gautam (2015) introduced Quasi-partial b-metric spaces and some related fixed point theorems. Beg and Pathak (2018) gave a generalised notion of weak partial metric space. Finally, Kanwal *et al.* (2019) presented a generalised concept from weak partial metric space to weak partial b-metric space.

Another effort of this kind is essentially due to Wardowski (2012), wherein the idea of F-contraction initiated. Piri and Kumam (2014) gave a generalisation of the Banach contraction principle by extending Wardowski (2012) results by introducing the concept of F-Suzuki contraction and obtained some results. Batra *et al.* (2020) extended the above concept using F-Kannan mappings and proved some results in metric space. A unique idea of improving the Banach contraction principle is due to Samet *et al.* (2012), wherein authors introduced a good idea of admissible mappings enough to extend, generalise and improve many existing results and also to yield constructive results. Recently, Ali and Kamran (2016) proved fixed point results in metric space by combining α-admissible, control function and F-contraction to form a generalised contraction.

Likewise, Aamri and El Moutawakil (2002) initiated the concept of (E.A.) property. Sintunavarat and Kumam (2011) introduced the notion of common limit range (CLR) property for single-valued mappings, which completely accommodated the conditions of the closeness of the degrees of the detailed mappings and showed its superiority over the (E.A.) property due to Aamri and El Moutawakil (2002). Motivated by this fact, Imdad *et al.* (2014) established a common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric (semi-metric) spaces. Also, Imdad *et al.* (2015) established the joint common limit range notion and proved the common fixed point theorem for a pair of non-self mappings in metric space.

Gopal *et al.* (2017) specified the fundamental properties for a fixed point theorem, ensuring a common fixed point for suitable assumptions. Those assumptions are sufficient and include: conditions of commutativity, containment of ranges of mappings, continuity of at least one mapping or weaker notion, contractive, and all essential common fixed point theorem attempts to obtain or soften required values of one or more such conditions.

Some special conditions on the pairs of mappings like weakly compatible mappings, $(E.A)$-property, faintly compatible mappings, common limit range property (CLR), coincidentally idempotent and joint common limit range property $(JCLR)$ have been utilised in different proofs by the researchers. One can see, Popa (1999), Imdad *et al.*(2016), Popa and Patriciu (2016), Popa (2017), Gupta *et al.* (2020) and the references therein. Researchers in this domain aimed at weakening one or more of these conditions. The use of weak conditions of commutativity is to improve common fixed point theorems in analysis.

With similar motivation, the study of nonlinear differential equations, integral equations and fractional differential equations (Kilbas *et al.* (2006)) has been made extensively among several nonlinear equations. In recent years, the theory of fixed points has led to the detailed study of these equations. Ran and Reurings (2004) obtained a variant of the Banach contraction principle for continuous monotone mappings in ordered metric spaces and also presented some applications to a system of nonlinear matrix equations. Afterwards, Nieto and Rodrigurz-Lopez (2005, 2007) slightly improved Ran-Reuring's fixed point theorem for monotone mappings (not necessarily continuous) besides presenting some applications to ordinary differential equations. For the recent advances, the reader referred to Harjani and Sadarangani (2010), Yan *et al.* (2012), Gupta *et al.* (2017), X *et al.* (2019), Borisut *et al.* (2019), Qawaqneh *et al.*(2019) and the references therein.

2.2 Fixed-point Theorems for Single-valued Mappings

In this section, we have made an attempt to present the introductory concepts and a brief survey about the development of fixed point theory. The track of the progress of fixed point theory has been indicated. Also some central results of great importance which have relevance to the development of the fixed point theory as well as to our investigations are listed.

Further, this section deals with contractive mappings, which are special types of continuous functions defined on metric spaces. Contractive mapping plays an important role in solving different applied mathematics problems, e.g., integral equations, differential equations, matrix equations and fractional differential equations.

Following the discovery of Banach contraction principle, researchers have found the ways to determine the fixed points of the maps by changing one or more conditions such as contractive condition, continuity of the maps and completeness of the space etc. Connell (1959) gave an example of a metric space X that is not complete where every contraction on X has a fixed point.

Kannan (1968) used the concept due to Connell (1959), to provide an alternative contractive condition different from the Banach contraction condition. As a result, the Kannan mapping is not forced to be continuous on complete metric spaces. Kannan (1968) used this new contractive condition and proved the following theorem for self mappings in complete metric spaces due to a generalisation of the Banach fixed point theorem.

Theorem 2.1 *(Kannan 1968) Let (X,d) be a complete metric space and a self mapping $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx,Ty) \leq k\{d(x,Tx)+d(y,Ty)\}, \tag{2.1}$$

for all $x,y \in X$ and $0 \leq k \leq \frac{1}{2}$. Then T has a unique fixed point $z \in X$ and for any $x \in X$ the sequence of iterate $\{T^n x\}$ converges to z.

An equivalent form of (2.1)

$$d(Tx,Ty) \leq \frac{k}{2}\{d(x,Tx)+d(y,Ty)\}, \tag{2.2}$$

for some $k \in (0,1)$.

Reich (1971) established the following result:

Theorem 2.2 (Reich 1971) Let X be a complete metric space with metric d, and let $T : X \to X$ be a function with the following property:

$$d(T(x),T(y)) \leq ad(x,T(x)) + bd(y,T(y)) + cd(x,y), \tag{2.3}$$

for all $x,y \in X$ where a,b,c are nonnegative and satisfy $a+b+c < 1$. Then T has a unique fixed point.

Note that $a = b = 0$ yields Banach's fixed point theorem, while $a = b, c = 0$ yields Kannan's fixed point theorem.

Hardy-Rogers (1973) gave an interesting generalisation of Banach contraction principle and Reich (1971) as follows;

Theorem 2.3 (Hardy and Rogers 1973) Let X be a complete metric space with metric d, and let $T : X \to X$ be a function with the following property:

$$d(Tx,Ty) \leq Ad(x,Tx) + Bd(y,Ty) + Cd(x,Ty) + Ed(y,Tx) + Fd(x,y), \tag{2.4}$$

for all $x,y \in X$ where A,B,C,E,F are nonnegative and satisfy $A+B+C+E+F < 1$. Then T has a unique fixed point.

If $A+B+F < 1$, we obtain Reich's result. If $A+B < 1$, we obtain Kannan mappings.

A classical proof was given by Ćirić (1974) for self-mapping, which is called generalised Banach contraction mapping to quasi-contraction mapping. This theorem proved for contraction map on complete metric space, which is stated below:

Theorem 2.4 *(Ćirić 1974) Let* $T : X \longrightarrow X$ *be a mapping of a metric space* (X,d) *into itself. Assume the following conditions are satisfied:*

$$d(Tx,Ty) \quad \leq \quad q\max(d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)), \tag{2.5}$$

$\forall\, x,y \in X$ *and* $0 \leq q < 1$. *Then* T *has a fixed point.*

Also, Subrahmanyam (1975) proved the converse of Banach fixed-point theorem using Kannan mapping. Moreover, the assumption of continuity of the mapping and the compactness condition on metric space is required for the existence of a fixed point for a strict type Kannan contraction.

Rhoades (1977) gave 250 different types of contractive conditions and obtained relationship amongst them which extends several ones from the literature. By using Rhoade's concept, the generalisation of Ćirić (1974) contraction can be done in the following form:

$$d(Tx,Ty) \quad \leq \quad \lambda \max M_T(x,y),$$

where

$$M_T(x,y) \quad = \quad \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2} \right\},$$

$\forall x,y \in X$. In this thesis, we utilise the same setting in generalisation of various types of mappings in abstract spaces.

Another way of generalising the Banach contraction principle is by using coincidence fixed point theorems. Coincidence fixed point theorems concern two functions S,T from set X into another set Y that, under certain conditions, admit a coincidence point. A coincidence point is an element $x \in X$ such that its images under the functions S,T are the same: in other words, $Sx = Tx$. Fixed point theorem consider one function S from a set X into itself and give conditions for the existence of a fixed point, that is, an element $x \in X$ such that $Sx = x$.

Jungck (1976) proved a fixed point theorem by using commutative maps to generalise the Banach contraction principle, which is as follows:

Theorem 2.5 *(Jungck 1976)*
A continuous self mapping f of a complete metric space (X, d) has a fixed point if and only if there exists a mapping $g : X \longrightarrow X$ which commutes with f and such that $g(x) \subseteq f(x)$,

$$d(gx,gy) \quad \leq \quad \lambda d(fx,fy), \forall x,y \in X, \tag{2.6}$$

where $0 \leq \lambda < 1$, furthermore, f and g have a unique common fixed point.

Sessa (1982) introduced the notion of the weak commutativity and relaxed it (point to point mapping) to smaller subsets of the domain of mappings. However, due to the reason that for commuting mapping, the proof of fixed point does not characterise the metric completeness, he established the idea of weak commutativity for two self-mapping S and T of a metric space (X,d). Jungck (1986) extended the concept of weak commutativity of mappings to compatible mappings because two similar elementary functions are not weakly commutative. It was desirable to introduce a less restrictive concept called compatibility, which requires that T and I commute on the potentially large set by replacing the identity map with a continuous map on a compatible map as follows:

Definition 2.1 *(Jungck 1986)*
A pair of self-mappings (T,I) of a metric space (X,d) is said to be compatible if

$$\lim_{n\to\infty} d(TIx_n, ITx_n) = 0,$$

wherever, $\{x_n\}$ is a sequence in X and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t, \text{ for some } t \in X.$$

We will also use the following definitions in chapter 3 and 6 to prove some common (coincidence) fixed point theorems for sequentially self-mappings in generalised metric spaces.

Definition 2.2 *(Jungck 1986) Let (S,T) be a pair of self mappings on a metric space (X,d). Then coincidence point of the pair (S,T) is a point $x \in X$ such that $Sx = Tx = x^*$, then x^* is called coincidence point of the pair (S,T). If $x^* = x$, then x is said to be a common fixed point.*

Definition 2.3 *(Sessa 1982, Jungck 1986) Let (S,T) be a pair of self mappings on a metric space (X,d). Then the pair (S,T) is said to be:*

(i) *Commuting if, for all $x \in X$, $S(Tx) = T(Sx)$,*

(ii) *Weakly commuting if, for all $d(S(Tx), T(Sx)) \leq d(Sx,Tx)$,*

(iii) *Compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever x_n is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$,*

(iv) *Weakly compatible if, for all $S(Tx) = T(Sx)$, for every coincidence point $x \in X$.*

The study for the applications of nonlinear equations in abstract spaces was initiated by Ran and Reurings (2004). The following theorem was proved by Ran and Reurings (2004) as a generalisation of the Banach contraction principle.

Theorem 2.6 *(Ran and Reurings 2004) Let X be a partially ordered set such that every pair $x, y \in X$ has a lower bound and an upper bound. Furthermore, let d be a metric on X such that (X,d) is a complete metric space. If T is a continuous, monotone (i.e., either order-preserving or order-reversing) map from X into X such that*

(i) $\exists 0 < c < 1 : d(Tx,Ty) \leq cd(x,y), \forall x > y,$

(ii) $\exists x_0 \in X : x_0 \leq Tx_0$ or $x_0 \geq Tx_0,$

then T has a unique fixed point $\bar{x}$. Moreover, for every $x \in X$,

$$\lim_{n\to\infty} T\bar{x} \quad = \quad \bar{x}.$$

Later, Suzuki (2008) proved two fixed point theorems, one of which is a new type of generalisation of the Banach contraction principle and does characterise the metric completeness. Furthermore, Rida *et al.* (2020) gave the generalisation of the Banach contraction principle that characterises metric completeness.

Another noted attempt to extend Banach contraction principle is essentially due to (Wardowski 2012).

The following explanations for developing the F-contraction definition were obtained from Wardowski (2012), Wardowski and Van Dung (2014), and Cosentino *et al.* (2015).

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

(F4) for each sequence $\{\alpha_n\} \subset \mathbb{R}^+$ of positive numbers such that

$$\tau + F(s^n \alpha_n) \leq F(\alpha_{n-1}),$$

for all $n \in \mathbb{N}$, $s \geq 1$ and some $\tau \in \mathbb{R}^+$, then

$$\tau + F(s^n \alpha_n) \leq F(s^{n-1} \alpha_{n-1}),$$

for all $n \in \mathbb{N}$.

We denote the family of all functions F satisfying conditions $((F1)-(F4))$ by $\mathscr{F}$. Some examples of functions $F \in \mathscr{F}$ are:

(1) $F_1(a) = \ln a \implies \frac{d(Tx,Ty)}{d(x,y)} \leq e^{-\tau}$;

(2) $F_2(a) = a + \ln a \implies \frac{d(Tx,Ty)}{d(x,y)} \leq e^{-\tau + d(x,y) - d(Tx,Ty)}$.

(3) $F_3(a) = -\frac{1}{\sqrt{a}} \implies \frac{d(Tx,Ty)}{d(x,y)} \leq \frac{1}{(1+\tau\sqrt{d(x,y)})^2}$

(4) $F_4(a) = \ln(a^2 + a) \implies \frac{d(Tx,Ty)(1+d(Tx,Ty))}{d(x,y)(1+d(x,y))} \leq e^{-\tau}$.

Remark 2.1 *(Rashwan and Hammad 2017)* For $p > 0$ and $a > 0$, then the function

(5) $F_5(a) = -\frac{1}{\sqrt[p]{a}} = -\frac{1}{a^{\frac{p}{2}}} \implies \frac{d(Tx,Ty)}{d(x,y)} \leq \frac{1}{\left(1+\tau\sqrt[p]{d(x,y)}\right)^{2-p}}$

belongs to $\mathscr{F}$.

Wardowski (2012) introduced a generalisation of the Banach contraction principle in metric spaces.

Definition 2.4 *(Wardowski 2012)* Let (X,d) be a metric space. A self-mapping T on X is called an F-contraction mapping if there exists $F \in \mathscr{F}$ and $\tau \in \mathbb{R}^+$ such that for all $x, y \in X$,

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)).$$

Wardowski (2012) proved the following fixed point theorem:

Theorem 2.7 *(Wardowski 2012) Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction mapping. If there exists $\tau > 0$ such that for all $x, y \in X$, $d(Tx,Ty) > 0$, implies*

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{2.7}$$

then T has a unique fixed point.

Since then, several researchers have done their research on F-contraction mappings for single and multivalued mappings in various spaces. For more literature, one can consult Consentino and Vetro (2014), Minak *et al.* (2014), Paesano and Vetro (2014), Altun *et* *al.* (2015), Durmaz and Altun (2016), Secelean (2016), Goswami *et al.* (2019), Lukacs and Kajanto (2018) and the references therein.

Secelean (2013) gave the following Lemma and definition as the generalisation of Wardowski (2012) notions.

Lemma 2.1 *(Secelean 2013) Let* $F : \mathbb{R}^+ \to \mathbb{R}$ *be an increasing function and* $\{x_n\}$ *be a sequence of positive real numbers. Then the following hold:*

(a) *if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$*, then* $\lim_{n\to\infty} \alpha_n = 0$*;*

(b) *if* $\inf F = -\infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$*, then* $\lim_{n\to\infty} F(\alpha_n) = -\infty$*.*

Let $\mathfrak{F}$ be the set of all functions defined as $F : \mathbb{R}^+ \to \mathbb{R}$, which satisfy the following conditions:

(F1) F is strictly increasing i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta \implies F(\alpha) < F(\beta)$;

(F2") there is a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$ or $\inf F = -\infty$;

(F3") F is continuous on $(0, \infty)$.

The following function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathfrak{F}$;

(i) $F_1(z) = \ln z$.

(ii) $F_2(z) = -\frac{1}{z}$;

(iii) $F_3(z) = -\frac{1}{z} + z$;

Piri and Kumam (2014), extended the results of Wardowski (2012) by introducing the concept of F-Suzuki contraction and obtained some interesting results, using the concept from Secelean (2013) as follows:

Definition 2.5 *(Piri and Kumam 2014) Let* (X,d) *be a metric space. A mapping* $T : X \longrightarrow X$ *is said to be an* F*-Suzuki contraction if there exists* $\tau > 0$*, such that for all* $x,y \in X$ *with* $Tx \neq Ty$

$$\frac{1}{2}d(x,Tx) < d(x,y) \implies \tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

where $F \in \mathfrak{F}$.

Piri and Kumam (2014) established a generalization of Banach contraction principle, which is as follows:

Theorem 2.8 *(Piri and Kumam 2014) Let* (X,d) *be a complete metric space and* $T : X \longrightarrow X$ *be an* F*-Suzuki contraction. Then* T *has a unique fixed point* $x^* \in X$ *and for every* $x_0 \in X$*, a sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *is convergent to* x^*.

Remark 2.2 *(Piri and Kumam 2014) We denote by* $\mathfrak{F}$ *the set of all functions satisfying* F*-suzuki type contraction condition due to (Secelean 2016, Piri and Kumam 2014) and denote by* $\mathscr{F}$ *the set of all functions satisfying* F*-contraction condition by (Wardowski 2012), then*

(i) $\mathscr{F} \not\subseteq \mathfrak{F}$,

(ii) $\mathfrak{F} \not\subseteq \mathscr{F}$,

(iii) $\mathscr{F} \cap \mathfrak{F} \neq \emptyset$.

For more details on F-Suzuki contraction mappings, one can see Alsulami *et al.* (2015), Budhia *et al.* (2016), Chandok *et al.* (2018) and the references therein.

Gopal *et al.* (2017) specified the fundamental properties for a fixed point theorem which ensures the existence of a common fixed point for suitable assumptions. Those assumptions are sufficient and include: conditions of commutativity, containment of ranges of mappings, continuity of at least one mapping or weaker notion, contractive, and all essential common fixed point theorem attempts to obtain or soften required values of one or more such conditions.

Goswami *et al.* (2019) defined F-contractive type mappings in b-metric spaces and proved some fixed point results with suitable examples. Recently, Batra *et al.* (2020) noticed that the definition introduced by Goswami *et al.* (2019) is not meaningful in general. Therefore, they provided suitable examples to support their opinion on this definition. Therefore, Batra *et al.* (2020) gave the notions of F-contraction and Kannan mapping to define a new class of contractions called F-Kannan mappings which is, in a true sense, a generalisation of Kannan mappings.

Motivated by Batra *et al.* (2020), we use the following notations: Let X be a non empty set and (X,d) denotes the metric space with metric d. Define the cardinality of a set A by $card\{A\}$ and $FixT$ denotes the set of all fixed points of a mapping T.

Batra *et al.* (2020) gave a new generalization family of contraction called F-Kannan mapping and introduced the following definition:

Definition 2.6 *(Batra et al. 2020) Let F be a mapping satisfying $(F1)-(F3)$. A mapping $T : X \to X$ is said to be an F-Kannan mapping if the following hold:*

(K1)

$$Tx \neq Ty \implies Tx \neq x \text{ or } Ty \neq y. \tag{2.8}$$

(K2) $\exists\, \Upsilon > 0$ *such that*

$$\Upsilon + F(d(Tx,Ty)) \leq F\left[\frac{d(x,Tx)+d(y,Ty)}{2}\right] \tag{2.9}$$

for all $x,y \in X$, with $Tx \neq Ty$.

The remark presented below is due to Batra *et al.* (2020).

Remark 2.3 (Batra *et al.* 2020) By properties of F, it follows that every F-Kannan mapping T on a metric space (X,d), satisfies following condition:

$$d(Tx,Ty)) \leq \frac{d(x,Tx)+d(y,Ty)}{2},$$

for every $x,y \in X$.

Further, it is concluded that $Card\{Fix\ T\} \leq 1$. Let T be a self map on a metric space (X,d). Following Picard (1890), if T has a unique fixed point $x^* \in X$ and $\lim\limits_{n\to\infty} T^n x = x^*$ for each $x \in X$, then T is called a Picard Operator (PO).

Then, the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the condition $(F1)-(F3)$ is denoted by $\mathscr{F}$.

One can use the following examples in Batra *et al.* (2020) of such functions $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ which satisfy $(F1)-(F3)$:

Example 2.1 (Batra *et al.* 2020) Let $F_1 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F_1(z) = \ln(z)$. Then clearly, $(F1) - (F3)$ are satisfied by $F_1(z)$. In fact $(F3)$ holds for every $k \in (0,1)$. Indeed,

$$d(Tx,Ty) \le e^{-\Upsilon}\left[\frac{d(x,Tx)+d(y,Ty)}{2}\right] \tag{2.10}$$

for all $x,y \in X$ with $Tx \neq Ty$.

Thus, $T : X \to X$ is a Kannan mapping with constant $k \in (0,1)$ satisfying

$$d(Tx,Ty) \le k\left[\frac{d(x,Tx)+d(y,Ty)}{2}\right] \tag{2.11}$$

for every $x,y \in X$.

Example 2.2 (Batra *et al.* 2020) Let $F_2 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F_2(z) = \ln(z) + z, z > 0$. Then, $(F1) - (F3)$ are satisfied by $F_2(z)$.

$$\frac{d(Tx,Ty)}{(d(x,Tx)+d(y,Ty))/2} e^{d(Tx,Ty) - \frac{d(x,Tx)+d(y,Ty)}{2}} \le e^{-\Upsilon}. \tag{2.12}$$

for all $x,y \in X$ with $Tx \neq Ty$.

The following lemma was proved by Batra *et al.* (2020).

Lemma 2.2 *(Batra et al. 2020) Let (X,d) be a metric space and $T : X \to X$ be an F-Kannan mapping. Then $d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$ for all $x \in X$.*

Batra *et al.* (2020) introduced an F-Kannan mapping using the properties by Subrahmanyam (1975) which is an extension of Goswami *et al.* (2019) and Wardowski (2012) results. They proved the following result.

Theorem 2.9 *(Batra et al.) 2020) Let (X,d) be a complete metric space and suppose $T : X \longrightarrow X$ is a F-Kannan mapping, then T is an Picard operator (PO).*

2.3 Some Types of Abstract Spaces

Now, we will introduce the abstract spaces on which our work is based.

The most natural and much-discussed fixed point theory and Banach contraction principle in metric spaces have been generalised and improved by introducing several variants such as metric spaces (Fréchet 1906), b-metric space (Bakhtin 1989, Czerwik 1993), partial metric space (Matthew 1999), generalised metric space (Branciari 2000), G-metric space (Mustafa and Sim 2006), cone metric space (Huang and Zhang 2007), Beg *et al.* 2009), S-metric space (Sedghi *et al.* 2012), metric-like space (Amini-Harandi 2012), b- metric-like space, partial b-metric space (Shukla 2014), quasi-S_b-metric space (Gupta and Gautam 2015), weak partial b-metric space (Kanwal *et al.* 2019) and several others.

2.3.1 Metric spaces

This subsection includes some important results about metric spaces.

Fréchet (1906) introduced the notion of distance between two points in an abstract set. This concept is known as metric space. The metric space is defined as follows:

Definition 2.7 *(Fréchet 1906) Let* (X,d) *be a metric space, where* X *is a nonempty set and* d *is a metric on* X *(or a distance function on* X*) that is a functional defined on* $X \times X$ *such that for all* $x,y,z \in X$*, we have*

(i) $d(x,y) \geq 0$, *(real valued, finite and non-negative),*

(ii) $d(x,y)= 0, \iff x = y,$

(iii) $d(x,y) = d(y,x)$, *(symmetric property),*

(iii) $d(x,y) \leq d(x,z) + d(y,z)$, *(triangle inequality).*

An abstract space is a class or set of elements of homogeneous nature but of any kind, in which the notion of proximity has been defined among such elements (Fréchet 1951). Thus, an abstract space is defined as an interaction of elements in the abstract set. Such elements are sets, sequences, functions and all types of numbers.

2.3.2 b-metric spaces

In this subsection, we describe the b-metric space and some of its properties.

The concept of metric space was generalised by Bakhtin (1989) and Czerwik (1993) to b-metric space due to some problems, especially the issue of convergence of measurable functions to a measure led to a generalisation of a metric's notion. Czerwik (1993) established b-metric spaces by introducing a parameter $s \geq 1$ in the triangle inequality as a coefficient and generalised Banach's contraction principle to these spaces.

Definition 2.8 (Czerwik 1993) Let X be a non empty set and $s \geq 1$ be a given real number. A function $d_b : X \times X \to [0,\infty)$ is called a b-metric on X if the following conditions are satisfied:

(B1) $d_b(x,y) = 0$ iff $x = y$;

(B2) $d_b(x,y) = d_b(y,x)$ and

(B3) $d_b(x,y) \leq s[d_b(x,z) + d_b(z,y)]$, for all $x,y,z \in X$.

Then, d_b is said to be b-metric, the pair (X,d_b,s) is called a b-metric space. The number $s \geq 1$ is called the parameter of (X,d_b,s).

The following are some examples that satisfy b-metric space axioms:

Example 2.3 *(Bakhtin 1989, Czerwik 1993) Consider the set* $X = [0,1]$ *endowed with the function* $d_b : X \times X \to [0,\infty)$ *defined by* $d_b(x,y) = |x-y|^2$ *for all* $x,y \in X$. *Thus* $(X,d_b,2)$ *is a b-metric space with parameter* $s = 2$, *but it is not metric space since* $s \neq 1$.

Example 2.4 (Chifu and Petrusel 2017) Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}_+$, $d_b(x,y) = |x-y|^3$. Then (X,d_b) is a b-metric space with constant $s = 3$.

Example 2.5 (Bota *et al.* 2011) Let $p \in (0,1)$, and Let

$$X \;=\; l_p(\mathbb{R}) := \Big\{x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\},$$

together with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R})$

$$d(x,y) \;=\; \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}}.$$

where $x = x_n, y = y_n$. Then (X,d) is a b-metric space with $s = 2^{\frac{1}{p}}$.

Definition 2.9 *(Bakhtin 1989, Czerwik 1993) Let* (X,d_d,s) *be a b-metric space. Let* $\{x_n\}$ *be a sequence in* X. *Then,*

(i) *a sequence* $\{x_n\} \subseteq X$ *converges to a point* $x \in X$ *if and only if*

$$\lim_{n \to +\infty} d_b(x_n,x) = 0.$$

(ii) *a sequence* $\{x_n\} \subseteq X$ *is called a* d_b*-Cauchy sequence if for each* $\varepsilon > 0$, *there exists some* $n(\varepsilon) \in \mathbb{N}$ *such that* $d_b(x_n,x_m) < \varepsilon$ *for all* $m,n \geq n(\varepsilon)$.

(iii) *a b-metric space* (X,d_b,s) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *is convergent in* X.

We take note of the following lemmas:
Lemma 2.3 (Dukic et al. 2011). Let Let (X,d_b,s) be a b- metric space with $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergence sequences to x and y, respectively. Then,

$$\frac{1}{s}d_b(x,y) \quad \leq \quad \lim_{n\to+\infty}\inf d_b(x_n,y_n) \leq \lim_{n\to+\infty}\sup d_b(x_n,y_n) \leq s^2 d_b(x,y).$$

In case $x=y$, we get

$$\lim_{n\to+\infty} d_b(x_n,y_n) \quad = \quad 0.$$

Moroever for each $x \in X$,

$$\frac{1}{s}d_b(x,y) \quad \leq \quad \lim_{n\to+\infty}\inf d_b(x_n,y) \leq \lim_{n\to+\infty}\sup d_b(x_n,y) \leq s^2 d_b(x,y).$$

Lemma 2.4 (Dukic *et al.* 2011, Radenovic *et al.* 2012). Let (X,d_b,s) be a b-metric space with $s \geq 1$. Suppose that $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to+\infty} d_b(x_n,x_{n+1}) \quad = \quad 0.$$

If $\{x_n\}$ is not a b-Cauchy sequence, then there exists $\varepsilon > 0$ and $\{x_{m_k}\}$ and $\{x_{n_k}\}$ two sequences of positive intergers such that

$$\begin{aligned}
\varepsilon \leq d_b(x_{m_k},x_{n_k}) &\leq \lim_{n\to+\infty}\inf d_b(x_{m_k},x_{n_k}) \leq \lim_{n\to+\infty}\sup d_b(x_{m_k},x_{n_k}) \leq s\varepsilon. \\
\frac{\varepsilon}{s} \leq d_b(x_{m_k},x_{n_{k+1}}) &\leq \lim_{n\to+\infty}\inf d_b(x_{m_k},x_{n_{k+1}}) \leq \lim_{n\to+\infty}\sup d_b(x_{m_k},x_{n_k+1}) \leq s^2\varepsilon. \\
\frac{\varepsilon}{s} \leq d_b(x_{m_{k+1}},x_{n_k}) &\leq \lim_{n\to+\infty}\inf d_b(x_{m_{k+1}},x_{n_k}) \leq \lim_{n\to+\infty}\sup d_b(x_{m_{k+1}},x_{n_k}) \leq s^2\varepsilon. \\
\frac{\varepsilon}{s^2} \leq d_b(x_{m_{k+1}},x_{n_{k+1}}) &\leq \lim_{n\to+\infty}\inf d_b(x_{m_{k+1}},x_{n_{k+1}}) \leq \lim_{n\to+\infty}\sup d_b(x_{m_{k+1}},x_{n_{k+1}}) \leq s^3\varepsilon.
\end{aligned}$$

Since then, several papers have been published on the fixed point theory of various classes of the single and multivalued maps in b-metric spaces. One can see Czerwik (1998), Czerwik (2001), Boriceanu *et al.* (2010), Kirk (2014), Roshan *et al.* (2016), Kamran *et al.* (2017), Chifu and Petrusel (2017), Kajanto and Lukacs (2018), Alamgir *et al.* (2019), Rasham *et al.* (2019b), Chifu and Karapinar (2020), Shoaib *et al.* (2020) and the references therein.

2.3.3 Partial metric spaces

Matthews (1994) introduced non-zero self-distance, which is widely applied in computer networking, data structure, and computer programming languages. The non-self distance generalises the metric to partial metric axioms, accommodating both metric and topological properties of abstract spaces. Some of these properties are complete spaces, Cauchy sequences and contraction fixed point theorem, which generalises the Banach contraction principle.

Further, Matthews (1994) replaced self distance by non-zero value to use it in computer semantics which is as follows:

Definition 2.10 *(Matthews 1994) A partial metric space is a pair* (X,p) *consisting of a non-empty set* X *together with a function* $p : X \times X \to \mathbb{R}^+$, *called the partial metric, such that for all* $x,y,z \in X$ *we have the following properties:*

(P1) $x = y$ *if and only if* $p(x,x) = p(x,y) = p(y,y)$*;*

(P2) $p(x,x) \leq p(x,y)$*;*

(P3) $p(x,y) \leq p(y,x)$*; and*

(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then, the pair (X,p) *is called a partial metric space.*

In partial metric space, it is not necessary that $p(x,x) = 0$, for every $x = y$, while in metric if $x = y$, then $p(x,x) = 0$. One can see important notions about partial metric spaces in Oltra and Valero (2004), Oltra *et al.* (2002), Shahzard and Valero (2013).

Let (X,p) be a partial metric space. Then, the p-open ball, with center $x \in X$ and radius $\varepsilon > 0$, is defined by: $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$.

Similarly, the p-closed ball, with center $x \in X$ and radius $\varepsilon > 0$, is defined by: $B_p[x,\varepsilon] = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$.

The family of p-open balls for all $x \in X$ *and* $\varepsilon > 0$, $U_p = B_p(x,\varepsilon) : x \in X, \varepsilon > 0$, the pair $(\mathbb{R}^+, p)$, where $p : X \times X \to \mathbb{R}^+$ is defined as $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$, is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has a base being the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where
$U_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ with respect to τ_p if and only if

$$\lim_{n\to\infty} p(x,x_n) = p(x,x).$$

Lemma 2.5 *(Matthews 1994) If p is a partial metric on X, then the function* $p^s : X \times X \longrightarrow \mathbb{R}$ *given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

for all $x,y \in X$*, is a metric on X.*

Definition 2.11 *(Matthews 1994) Let (X, p) be a partial metric space, then:*

(i) *a sequence* $\{x_n\}$ *in a partial metric space* (X,p) *is called a p-Cauchy sequence if only if* $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ *exists and is finite.*

(ii) *a sequence* $\{x_n\}$ *is a Cauchy sequence in* (X,p) *if and only if it is a p-Cauchy sequence in a metric* (x,p^s).

(iii) *a partial metric space* (X,p) *is said to be p-complete if every p-Cauchy sequence* $\{x_n\}$ *in X is p- convergent, with respect to* τ_p*, to a point* $x \in X$ *such that*

$$\lim_{n,m\to\infty} p(x_n,x_m) = p(x,x).$$

The success of partial metric in computer science lies in the fact that every partial metric p induces a partial order $\leq_p$ on $X(x \leq_p y \Leftrightarrow p(x,y) = p(x,x))$, in such a way that increasing sequence of elements have a supremum with respect to $\leq_p$ and converges to it with respect to the partial metric topology τ_p. In recent years, several researchers extended metrical fixed point theorems to partial metric spaces see, Rus (2008), Shahzard and Valero (2013), Karapinar *et al.* (2019) and the references therein.

2.3.4 Generalised metric spaces

This subsection explains a generalised metric space concept used in Chapter 3 to prove a coincidence point using F-Kannan mapping.

Branciari (2000) introduced a class of generalised metric spaces by replacing triangular inequality with similar ones which involve four or more points instead of three and improved the Banach contraction mapping principle. Azam and Arshad (2008), using the concept of Branciari (2000), investigated fixed points for the mappings given by Kannan (1968) by applying the rectangular property in a generalised metric space.

Definition 2.12 *(Azam and Arshad 2008) Let X be a non-empty set. Suppose that the mapping* $d : X \times X \to \mathbb{R}$, *satisfies:*

(i) $d(x,y) \geq 0$, *for all* $x,y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;

(ii) $d(x,y) = d(y,x)$, *for all* $x,y \in X$;

(iii) $d(x,y) \leq d(x,w) + d(w,z) + d(z,y)$, *for all* $x,y \in X$ *and for all distinct points* $w,z \in X - \{x,y\}$ *[rectangular property].*

Then d is called a generalised metric and (X,d) *is a generalised metric space.*

This motivated several researchers to prove fixed point results in such spaces. For more detail on fixed point theory of generalised metric spaces, we refer the reader to Sarma *et al.* (2009), Samet (2010), Aydi *et al.* (2012), Shatanawi *et al.* (2012), Aydi *et al.* (2019) and Souayah *et al.*(2019).

2.3.5 *D*-metric spaces

Dhage (2000) introduced a new class of generalised metric spaces called *D*-metric spaces, which is as follows:

Definition 2.13 (Dhage 2000) A function $D : X \times X \times X \to \mathbb{R}$ is said to be a *D*-metric on X if it satisfies the following properties:

(i) $D(x,y,z) \geq 0$ for all $x,y,z \in X$ and equality holds if and only if $x = y = z$ (non-negativity);

(ii) $D(x,y,z) = D(x,z,y) = (symmetry)$;

(iii) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x,y,z \in X$ (tetrahedral inequality).

A nonempty set X together with a *D*-metric is called a *D*-metric space and is denoted by (X,D).

Using this definition, Dhage (2000) proved the existence of a unique fixed point for self-mapping satisfying a contractive condition.

On the other hand, the existence of fixed points in D-metric spaces has been considered by Rhoades (1996) to prove two fixed point theorems for the generalised metric spaces. Dhage (1999) and Dhage *et al.* (2000) proved the results for a common fixed point principle in D-metric spaces. Ahmad *et al.* (2001) introduced a fixed point theorem for expansive mappings in D-metric spaces. Singh and Sharma (2002) and Sedghi *et al.* (2007) proved common fixed point theorems under compatible and weakly commuting mappings in D-metric spaces.

2.3.6 G-metric spaces

This subsection discusses the G-metric space concept, which will be applied in chapter 5 to prove fixed point theorems in these spaces.

Mustafa and Sims (2006) gave a generalisation of D-metric space to G-metric space soon after identifying some shortcomings concerning the fundamental topological structure on D-metric spaces. They defined several notions, such as continuity, completeness, compactness, convergence, and product space in the G-metric space setting. In doing so, they replaced the tetrahedral inequality with an inequality involving the repetition of indices.

Mustafa and Sims (2006) formulated the axioms of G-metric spaces as follows:
Definition 2.14 *(Mustafa and Sims 2006) Let X be a non empty set and* $G : X \times X \times X \to \mathbb{R}_+$ *be a function satisfying the following conditions:*

(G1) $G(x,y,z) = 0$ *if* $x = y = z$;

(G2) $0 < G(x,x,y)$, *for all* $x,y \in X$ *with* $z \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y \in X$ *with* $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ *(symmetry in all three variables);*

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, *for all* $x,y,z,a \in X$ *(rectangle inequality).*

The function G is called a generalised metric or G-metric, and the pair (X,G) *is called a G-metric space.*

The following example satisfies the above axioms:

Example 2.6 (Mustafa and Sims 2006) Let $X = \mathbb{R}$ be the set of all real number. Define $G : \mathbb{R}^3 \to \mathbb{R}^+$ by

$$G(x,y,z) = |x-y| + |y-z| + |x-z|,$$

for all $x,y,z \in X$. Then it is clear that (X,G) is a G-metric space with a G-metric on X.

Note that if $G(x,y,z) = 0$ then $x = y = z$.

Mustafa and Sims (2006) proved the following proposition satisfying a G-metric properties.

Proposition 2.1 (Mustafa and Sims 2006) Let (X,G) be a G-metric space, then the metric associated with G satisfies:

(i) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,

(ii) $G(x,y,y) \leq 2G(y,x,x)$,

(iii) $G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$,

for all $x,y,z,a \in X$.

Mustafa and Sims (2006) established some topological properties such as convergence, completeness and continuity in G-metric spaces as follows:

Definition 2.15 (Mustafa and Sims 2006) Let (X,G) be a G-metric space. A sequence $x_n \in X$ is said to be:

(i) G-convergent to $x \in X$ if for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x,x_n,x_n) < \varepsilon$ for all $n \geq k$;

(ii) a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n,x_m,x_p) < \varepsilon$ for all $n,m,p \geq n_0$, i.e., $G(x_n,x_m,x_p) \to 0$ as $n,m,p \to \infty$.

Definition 2.16 (Mustafa and Sims 2006) A G-metric space is said to be G-complete if every G-Cauchy sequence in X is G-convergent. Every G-metric on X defines a metric

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \tag{2.13}$$

for all $x,y \in X$.

Proposition 2.2 (Mustafa and Sims 2006) Let (X,G) be a G-metric space. Then the following properties are equivalent:

(1) (x_n) is G-convergent to x;

(2) $G(x_n, x_m, x_n) \to 0$ as $n, m \to \infty$;

(3) $G(x_n, x, x) \to 0$ as $n \to \infty$;

(4) (x_n) is a G-Cauchy sequence;

(5) For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for $n, m > k$.

Definition 2.17 (Mustafa and Sims 2006) Let (X, G) and (X', G') be two G-metric spaces and let $f : (X, G) \to (X', G')$. Then, the map f is said to be G-continuous at $x \in X$ if for $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ and $G(a, x, y) < \delta$, we have $G'(fa, fx, fy) < \varepsilon$. The function f is G-continuous if it is G-continuous for each $a \in X$.

Proposition 2.3 *(Mustafa and Sims 2006)* Let (X, G) and (X', G') be two G-metric spaces and let $f : (X, G) \to (X', G')$. Then the map f is said to be G-continuous at $x \in X$ if and only if f is sequentially continuous, i.e., whenever (x_n) is G-convergent to x, the sequence fx_n is G-convergent to fx.

Lemma 2.6 (Mustafa and Sims 2006) Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is continuous in all it is variables.

Further, Mustafa *et al.* (2009) gave the existence of a fixed point for mapping in G-metric spaces, which influenced many other authors. Since then, several researchers have proved fixed point theorems in G-metric spaces. Chugh *et al.* (2010) proved the results of property P in G-metric spaces. Shushanta (2011) proved property P of Ciric operator on G-metric spaces. Rani *et al.* (2012) introduced the version of common fixed point theorems for compatible and weakly compatible maps in G-metric spaces. Jleli and Samet (2012) gave some remarkable results on G-metric spaces. Moreover, Agarwal *et al.* (2013) gave a theorem on couple fixed point results in asymmetric G-metric spaces. For more details on G-metric spaces, one can see Mustafa and Obiedat (2010), Shatanawi and Abbas (2012), Azam and Mehmad (2013), Agarwal *et al.* (2015), Mustafa and Arshad (2017), Nagaraju (2020), Shoaib and Shahzad (2020) and the reference therein.

2.3.7 Cone metric spaces

Huang and Zhang (2007) introduced the structure of cone metric by replacing real numbers with an ordering Banach space and established a convergence criterion for sequences in a cone metric space to generalise the Banach fixed point theorem. Huang and Zhang (2007) proved some fixed point theorem for Kannan type contractive condition in normal cone metric space; however, Rezapour and Hamlbarani (2008) omitted this concept in some results by Huang. Several authors have investigated fixed point theorems and common fixed point theorems of self-mappings for normal and non-normal cones in cone metric spaces. We refer the reader to Ili and Rakoevi (2008), Abbas and Rhoades (2009), Sahin and Telci (2009), Radenovic (2009), Morales and Rojas (2010), Kadelburg *et al.* (2011), Vetro and Radenovic (2018) and the references therein.

Beg *et al.* (2009) studied common fixed points for a pair of maps on topological vector space (TVS) valued cone metric spaces by relaxing the normality conditions imposed by Huang and Zhang (2007). They showed that the class of TVS cone metric spaces is larger than the class of cone metric spaces, used in Azam and Beg (2013), Djordjevic *et al.* (2011), Kadelburg *et al.* (2010), Kadelburg *et al.* (2012), Radenovic and Rhoades (2009) and the references therein.

We will require the following definitions and preliminary results to prove our results. Let (E,τ) be always a topological space and P a subset of E. Then P is called a cone whenever:

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $ax+by \in P$ for all $x,y \in P$ and nonnegative real numbers a,b;

(iii) $P \cap (-P) = \{0\}$.

Let $P \subseteq E$ be a given cone. If the interior of P (int P), is non empty, we say that P is solid. If P is soilid cone, then P is a component of P, and in this case we use the notation $x \ll y$ to indicate that $y - x \in intP$. Note that if $x \ll y$ and $y \leq z$, then $x \ll z$ for all $x, y, z \in intP$.

The following are axioms that satisfy TVS-valued cone metric spaces:

Definition 2.18 *(Beg et al. 2009) Let X be a non-empty set. Suppose that the mapping $\rho : X \times X \to E$, satisfies:*

(i) $0 \leq \rho(x,y)$, *for all* $x, y \in X$ *and* $\rho(x,y) = 0$ *if and only if* $x = y$;

(ii) $\rho(x,y) = \rho(y,x)$, *for all* $x, y \in X$;

(iii) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$, *for all* $x, y \in X$ *and for all* $x, y, z \in X$.

Then ρ is called a cone metric on X and (X, ρ) is called a topological vector space valued cone metric space.

Example 2.7 *(Schaefer 1971, Morales and Rojas 2010, Azam and Beg 2013) Let $E = (C_{[0,1]}, \mathbb{R}^2), P = \{(x,y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $\rho : X \times X \to E$ such that $\rho(x,y) = |x-y|\psi(t)$, where $\psi(t) = e^t$. Then (X, ρ) is a TVS valued cone metric space.*

2.3.8 S-metric spaces

Sedghi *et al.* (2012) gave a fascinating generalization of *D*-metric space and *G*-metric space to S-metric space by formulating its properties as follows:

Definition 2.19 (Sedghi *et al.* 2012) Let X be a non empty set. An S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

(S1) $S(x,y,z) \geq 0$;

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$; and

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X, S) is called an S-metric space.

Some of the examples which satisfy the above axioms are:

(1) Let $X = \mathbb{R}^n$ and $||.||$ a norm on X, then,

$$S(x,y,z) = ||y + z - 2x|| + ||y - z||,$$

is an S-metric on X.

(2) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then,

$$S(x,y,z) = \|x-z\| + \|y-z\|,$$

is an S-metric on X.

(3) Let X be a non empty set, d is an ordinary metric on X, then

$$S(x,y,z) = d(x,z) + d(y,z),$$

is an S-metric on X.

Sedghi *et al.* (2012) proved that D-metric is S-metric, but in general the converse is not true.

We give some of vivid illustrative example on S-metric spaces as follows:

Example 1.1 (Sedghi *et al.* 2012) Let $X = \mathbb{R}^2$, d is an ordinary metric on X, therefore,

$$S(x,y,z) = d(x,y) + d(x,z) + d(y,z),$$

is an S-metric on X. If we connect the points x,y,z by a line we have a triangle and if we choose a point a mediating this triangle then the equality

$$S(x,y,z) \quad = \quad S(x,x,a) + S(y,y,a) + S(z,z,a),$$

holds. Indeed,

$$\begin{aligned} S(x,y,z) &= d(x;y) + d(x,z) + d(y,z) \\ &\leq d(x,a) + d(a,y) + d(x,a) + d(a,z) + d(y,a) + d(a,z) \\ &= S(x,x,a) + S(y,y,a) + S(z,z,a). \end{aligned}$$

Also, Sedghi *et al.* (2012) proved the following Lemmas to satisfy S-metric:
Lemma 2.7 (Sedghi *et al.* 2012) In an S-metric space, we have

$$S(x,x,y) = S(y,y,x).$$

Lemma 2.8 *(Sedghi and Van Dung 2014)* Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n \to x$ and $\lim_{n\to\infty} y_n \to y$, then

$$\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y).$$

Lemma 2.9 (Sedghi *et al.* 2018) Let (X,S) be an S-metric space. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} S(x_n,x_n,y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$, then $\lim_{n\to\infty} y_n = t$.

Definition 2.20 (Sedghi *et al.* 2012) Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows:

(B1) $B_S(x,r) = \{y \in X : S(y,y,x) < r\}$;

(B2) $B_S[x,r] = \{y \in X : S(y,y,x) \leq r\}$.

The topology induced by the S-metric is the topology generated by the base of all open balls in X.

Definition 2.21 (Sedghi *et al.* 2012) Let (X,S) be an S-metric space.

(i) A sequence $\{x_n\} \in X$ converges to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n \to x$.

(ii) A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \geq n_0$.

(iii) The S-metric space (X,S) is complete if every Cauchy sequence is a convergent sequence in X.

Several researchers gave different generalized results in S-metric spaces. For more details, one can see Sedghi and Van Dung (2014), Popa and Patriciu (2015), Chaipornjareansri (2016), Kim *et al.* (2016), Chaipornjareansri (2018), Ansari *et al.* (2019) and the references therein.

Sedghi and Van Dung (2014) gave a generalisation of Kannan contraction in S-metric space and proved the following result.

Theorem 2.10 *(Sedghi and Van Dung 2014) Let T be a self-mapping on a complete S-metric space (X,S) and*

$$S(Tx,Tx,Ty) \;\; \leq \;\; a(S(Tx,Tx,x)+S(Ty,Ty,y)),$$

for some $a \in [0,\frac{1}{2}]$ and all $x,y \in X$. Then T has a unique fixed point in X.

2.3.9 Metric-like spaces

Amini-Harandi (2012) generalised the concept of partial metric space by introducing the metric-like space. By this, we mean a pair (X,σ), where X is a nonempty set and $\sigma : X \times X \to \mathbb{R}$ satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$. This notion will be utilised in chapter 6 to prove common fixed point theorem.

Amini-Harandi (2012) gave a generalization of partial metric spaces to metric-like spaces by introducing the following properties:

Definition 2.22 *(Amini-Harandi 2012)* A metric-like space is a pair (X,σ) consisting of a non-empty set X together with a function $\sigma : X \times X \to \mathbb{R}^+$, called the partial metric, such that for all $x,y,z \in X$, the following conditions hold:

$(\sigma 1)$ $\sigma(x,y) = 0 \implies x = y$;

$(\sigma 2)$ $\sigma(x,y) = \sigma(y,x)$; and

$(\sigma 3)$ $\sigma(x,y) \leq \sigma(x,y) + \sigma(y,z)$.

Then σ is called a metric-like space on X, so a pair (X,σ) is called a metric-like space.

The metric-like on X satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$. The following are some characteristics of metric-like spaces:

Definition 2.23 *(Amini-Harandi 2012) Let (X,σ) be a metric-like space.*

(i) Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}, \forall x \in X \text{ and } \varepsilon > 0.$$

(ii) A sequence $\{x_n\}$ in metric-like space (X,σ) converges to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} \sigma(x_n,x) = \sigma(x,x).$$

(iii) The sequence $\{x_n\}_{n=0}^{\infty}$ *is said to be* σ*-Cauchy if the limit*

$$\lim_{n,m\longrightarrow\infty} \sigma(x_n,x_m)$$

exists and is finite.

(iv) The space (X,σ) *is called complete if for every* σ*-Cauchy sequence* $\{x_n\}_{n=0}^{\infty}$*, there exists some* $x \in X$ *such that*

$$\lim_{n\longrightarrow\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{n\longrightarrow\infty} \sigma(x_n,x_m).$$

(v) A sequence $\{x_n\}$ *in* (X,σ) *is said to be* 0-σ*-Cauchy sequence if* $\lim\limits_{n\longrightarrow\infty} \sigma(x_n,x_m) = 0$. *The space* (X,σ) *is said to be* 0-σ*- complete if every* 0-σ*-Cauchy sequence in* X *converges in* (τ_σ) *to a point* $x \in X$ *such that* $\sigma(x,x) = 0$.

(vi) A mapping $T : X \longrightarrow X$ *is continuous if the following limit exists (finite) and*

$$\lim_{n\longrightarrow\infty} \sigma(x_n,x) = \sigma(Tx,x).$$

Remark 2.4 *(Amini-Harandi 2012) It is easy to see that a metric space is a partial metric space and each partial metric space is a metric-like space, but the converse is not valid.*

Remark 2.5 *(Amini-Harandi 2012) Every partial metric space is a metric-like space. This can be illustrated by the use of the following example:*

Example 2.8 *(Amini-Harandi 2012) Let* $X = \{0,1\}$ *and* $\sigma : X \times X \longrightarrow \mathbb{R}^+$ *be defined by*

$$\sigma(x,y) = \begin{cases} 2, & if\ x = y = 0, \\ 1, & otherwise. \end{cases}$$

Then (X,σ) is a metric-like space, but is not a partial metric space since $\sigma(0,0) \neq \sigma(0,1)$, then (X,σ) is not a partial metric space.

Later, several authors gained interest in working in metric-like spaces, one can see Aydi *et al.* (2015), Aydi and Karapinar (2015), Khammahawong (2017), Joshi *et al.* (2017) and the references therein.

2.3.10 Partial b-metric spaces

Shukla (2014) gave an extension by combining partial metric space and b-metric space concepts to partial b-metric space.

Definition 2.24 *(Shukla 2014) A partial b-metric on a non-empty set* X *is a function* $b : X \times X \to \mathbb{R}^+$ *such that for all* $x,y,z \in X$*:*

(Pb1) $x = y$ *if and only if* $b(x,x) = b(x,y) = b(y,y)$,

(Pb2) $b(x,x) \leq b(x,y)$,

(Pb3) $b(x,y) = b(y,x)$ *and*

(Pb4) *there exist a real number* $s \geq 1$ *such that* $b(x,y) \leq s[b(x,z) + b(z,y)] - b(z,z)$.

A partial b-metric space is a pair (X,b) such that X is a non-empty set and b is a partial b-metric on X. The number $s \geq 1$ is called the coefficient of (X,b).

Mustafa *et al.* (2013) gave an extension of partial b-metric space as follows:

Definition 2.25 *(Mustafa et al. 2013) Let X be a non empty set and* $s \geq 1$ *be a given real number. A function* $p_b : X \times X \to \mathbb{R}^+$ *is called a partial b- metric if for all* $x,y,z \in X$ *the following conditions are satisfied:*

$(PB1)$ $x = y \Longleftrightarrow p_b(x,x) = p_b(x,y) = p_b(y,y)$,

$(PB2)$ $p_b(x,x) \leq p_b(x,y)$,

$(PB3)$ $p_b(x,y) = p_b(y,x)$ *and*

$(PB4)$ $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y) - p_b(z,z)] + \frac{1-s}{2}[p_b(x,x) + p_b(y,y)]$.

The pair (X, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

Example 2.9 *(Shukla 2014)* Let $X = \mathbb{R}^+$, $q > 1$ be a constant and $p_b : X \times X \to \mathbb{R}^+$ be defined by

$$p_b(x,y) = [\max\{x,y\}]^q + |x-y|^q,$$

for all $x, y \in X$. Then, (X, p_b) is a partial b-metric space with the coefficient $s = 2^q > 1$, but it is neither a b-metric nor a partial metric space.

2.3.11 Weak metric spaces

We start by introducing weak metric spaces, which will be important in developing our main results in chapter five and chapter six of this thesis.

Beg and Pathak (2018) gave a generalized notion of weak partial metric space as follows:

Definition 2.26 *(Beg and Pathak 2018) Let X be a non empty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on X if for all $x, y, z \in X$ the following conditions are satisfied:*

$(WP1)$ $q(x,x) = q(x,y) \Longleftrightarrow x = y$,

$(WP2)$ $q(x,x) \leq q(x,y)$,

$(WP3)$ $q(x,y) = q(y,x)$ *and*

$(WP4)$ $q(x,y) \leq q(x,z) + q(z,y)$.

The pair (X,q) is called a weak partial metric space.

Some examples of weak partial metric spaces are the following
Example 2.10 *(Beg and Pathak 2018)*

(1) $(\mathbb{R}^+, q)$, *where* $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$q(x,y) = |x-y| + 1,$$

for all $x, y \in \mathbb{R}^+$.

(2) $(\mathbb{R}^+, q)$, *where* $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$q(x,y) = \frac{1}{4}|x-y| + \max\{x,y\},$$

for all $x, y \in \mathbb{R}^+$.

(3) $(\mathbb{R}^+, q)$, *where* $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$q(x,y) = \max\{x,y\} + e^{|x-y|} + 1,$$

for all $x, y \in \mathbb{R}^+$.

Kanwal *et al.* (2019) gave a generalised concept from weak partial metric space to weak partial b-metric space as follows:
Definition 2.27 *(Kanwal et al. 2019) Let $M \neq \emptyset$ and $s \geq 1$, a function $\rho_b : M \times M \to \mathbb{R}^+$ is called a weak partial b-metric on M if for all $x, y, z \in M$, following conditions are satisfied:*

$(WPB1)$ $\rho_b(x,x) = \rho_b(x,y) \Longleftrightarrow x = y$,

$(WPB2)$ $\rho_b(x,x) \leq \rho_b(x,y)$,

$(WPB3)$ $\rho_b(x,y) = \rho_b(y,x)$ *and*

$(WPB4)$ $\rho_b(x,y) \leq s[\rho_b(x,z) + \rho_b(z,y)]$.

The pair (M,ρ_b) is called a weak partial b-metric space.

Some of the examples of weak partial b-metric space are the following:

Example 2.11 *(Kanwal et al. 2019)*

(1) $(\mathbb{R}^+,\rho_b)$, *where* $\rho_b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$\rho_b(x,y) = |x-y|^2 + 1,$$

for all $x,y \in \mathbb{R}^+$.

(2) $(\mathbb{R}^+,q)$, *where* $\rho_b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$\rho_b(x,y) = \frac{1}{2}|x-y|^2 + \max\{x,y\},$$

for all $x,y \in \mathbb{R}^+$.

Definition 2.28 *(Kanwal et al. 2019) A sequence* $\{x_n\}$ *in* (M,ρ_b) *is said to converge to a point* $x \in X$*, if and only if*

$$\rho_b(x,x) = \lim_{n\to\infty} \rho_b(x,x_n).$$

Definition 2.29 *(Kanwal et al. 2019) Let* (M,ρ_b) *be a weak partial b-metric space. Then*

(i) *A Cauchy sequence in metric space* (M,ρ_b^s) *is Cauchy in* M.

(ii) *If the metric space* (M,ρ_b^s) *is complete, so is weak partial b-metric space* (M,ρ_b).

Remark 2.6 *(Kanwal et al. 2019) If* ρ_b *is a weak partial b-metric on* M*, the function* $\rho_b^s : M \times M \to \mathbb{R}^+$ *given by*

$$\rho_b^s(x,y) = \rho_b(x,y) - \frac{1}{2}[\rho_b(x,x) + \rho_b(y,y)],$$

define a b metric on M*. Further, a sequence* $\{s_n,\}$ *in* (M,ρ_b^s) *converges to a point* $s \in M$*, iff*

$$\lim_{n,m\to\infty} \rho_b^s(x,y) = \lim_{n\to\infty} \rho_b(s_n,s) = \rho_b(s,s).$$

Another weak partial metric space introduced by Gupta and Gautam (2015). The property of quasi-partial b-metric space defined as follows:

Definition 2.30 *(Gupta and Gautam 2015) A quasi-partial b-metric space on a non empty set X is a mapping* $qp_b : X \times X \to \mathbb{R}^+$ *such that for some real number* $s \geq 1$ *and all* $u, v, z \in X$*:*

$(QPb1)$ $qp_b(u,u) = qp_b(u,v) = qp_b(v,v) \Rightarrow u = v;$

$(QPb2)$ $qp_b(u,u) \leq qp_b(u,v);$

$(QPb3)$ $qp_b(u,u) \leq qp_b(v,u);$ *and*

$(QPb4)$ $qp_b(u,v) \leq s[qp_b(u,z) + qp_b(v,z)] - qp_b(z,z).$

A quasi-partial b-metric space is a pair (X, qp_b) *such that X is a non-empty set and* (X, qp_b) *is a quasi partial b-metric on X. The number s is called the coefficient of* (X, qp_b)*.*

For a quasi-partial b-metric space (X, qp_b), the function $d_{qpb} : X \times X \to R^+$ defined by $d_{qp_b}(u,v) = qp_b(u,v) + qp_b(v,u) - qp_b(u,u) - qp_b(v,v)$ is a b-metric on X.

Lemma 2.10 *(Gupta and Gautam 2015) Every quasi-partial metric space is a quasi-partial b-metric space. However, the converse does not need to be true.*

Lemma 2.11 *(Gupta and Gautam 2015)* Let (X, qp_b) be a quasi-partial b-metric space and (X, d_{qp_b}) be the corresponding b-metric space. Then (X, d_{qp_b}) is complete if (X, qp_b) is complete.

An immediate example of quasi-partial b-metric space is given in Gupta and Gautam (2015) and Gautam and Verma (2021).

Motivated by the works of Czerwik (1993) and sedghi *et al.* (2012), Nizar and Nabil (2016) introduced the notion of S_b-metric space.

Definition 2.31 *(Nizar and Nabil 2016)* Let X be a non empty set and let $s \geq 1$ be a given number. A function $S_b : X \times X \times X \to [0, \infty)$ is said to be S_b-metric if and only if for all $u, v, z, t \in X$ the following conditions hold:

$(S1)$ $S_b(u,v,z) \geq 0;$

$(S2)$ $S_b(u,u,v) = S_b(v,v,u)$ for all $u, v \in X$; and

$(S3)$ $S_b(u,v,z) \leq s[S_b(u,u,t) + S_b(v,v,t) + S_b(z,z,t)].$

The pair (X, S_b) is called an S_b-metric space.

2.4 Multivalued Mapping Notions in Some Abstract Spaces

Pompeiu (1905) defined the concept of distance between two closed sets, in the context of complex analysis, in his PhD thesis. Later, Hausdorff (1914) considered all the basic concepts introduced by Pompeiu in his book *Grundzuge der Mengenlehre*, but in the general setting of a metric space, and adopted an alternative way to symmetrise the asymmetric distances $D(A,B)$ and $D(B,A)$, by defining what is currently denoted by $H(A,B)$ and commonly named Hausdorff metric. The Hausdorff metric notion deals with the measures of distance between two sets A and B of $CB(X)$.

We introduce some measures of Hausdorff metric in some spaces which will be considered in this thesis, specifically in chapter 4, to prove some results on multivalued contraction mappings:

2.4.1 Multivalued mappings in metric spaces

Nadler (1969) introduced multivalued contraction mappings using the Hausdorff metric and extended Banach's contraction principle (Banach 1922) from single-valued to multivalued mappings. Since then, several researchers have generalised these results for multivalued mappings in various spaces. Thus, the multivalued mappings theory has many applications in diverse areas, such as control theory, approximation theory, differential equations and economics.

Definition 2.32 *(Nadler 1969) Let (X,d) be a metric space. A map $T:X\to CB(X)$ is said to be a mult-valued contraction if there exists $0\leq\lambda<1$ such that*

$$H(Tx,Ty)\leq\lambda d(x,y), \tag{2.14}$$

for all $x,y\in X$.

We denote closed and bounded sets by $CB(X)$, which is the class of all non-empty closed and bounded subsets of X. Let $x\in X$ and $A\subset X$, then

$$d(x,A)=\inf\Big\{d(x,a):a\in A\Big\}.$$

For $A,B\in CB(X)$, the function $H:CB(X)\times CB(X)\to[0,\infty)$ such that

$$H(A,B)=\max\Big\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\Big\},$$

is said to be Hausdorff distance.

2.4.2 Multivalued mappings in b-metric spaces

This subsection gives the insights for Multivalued mappings in b-metric spaces, utilised in chapter 4, section 4.5, to prove some fixed point results.

We obtain the description and properties of the partial Hausdorff b-metric from Czerwik *et al.* (2001).

Let $CB^b(X)$ be the family of all non-empty, closed and bounded subsets of a b-metric space (X, d_b), induced by the b-metric d_b.

We state some properties of the partial Hausdorff metric H_b.

Lemma 2.12 *(Czerwik et al. 2001) Let (X, d_b, s) be a b-metric space. For all $A, B, C \in CB^b(X)$, and $x, y \in X$, we have*

(i) $d_b(x,B) \leq d_b(x,y)$;

(ii) $\delta_b(A,B) \leq H_b(A,B)$;

(iii) $d_b(x,B) \leq H_b(A,B)$ *for any* $x \in A$;

(iv) $H_b(A,A) = 0$;

(v) $H_b(A,B) = H_b(B,A)$;

(vi) $H_b(A,C) \leq s[H_b(A,B) + H_b(B,C)]$;

(vii) $d_b(x,A) \leq s[d_b(x,y) + d_b(y,A)]$.

Remark 2.7 (Czerwik *et al.* 2001) Let (X, d_b, s) be a b-metric space. The function $H_b : CB^b \times CB^b \to [0, \infty)$ is a generalized Hausdorff b-metric, that is

$$H_b(A,B) = \begin{cases} \max\{\delta_b(A,B), \delta_b(B,A)\}, & A \neq \emptyset \neq B, \\ 0, & A = \emptyset = B, \\ +\infty, & otherwise. \end{cases}$$

It is easy to see that $H_b(A,B) = 0 \to A = B$.

Lemma 2.13 (Czerwik *et al.* 2001) Let (X, d_b, s) be a b-metric space. For $A \in CB^b(X)$ and $x \in X$, we have

$$d_b(x,A) = 0 \Leftrightarrow x \in \bar{A} = A,$$

where $\bar{A}$ denotes the closure of A with respect to the b-metric d_b.

Lemma 2.14 (Czerwik *et al.* 2001) Let (X, d_b, s) be a b-metric space. For $A, B \in CB^b(X)$. Then for each $h > 1$ and for each $a \in A$ there exists $b \in B$ such that $d_b(a,b) \leq hH_b(A,B)$.

2.4.3 Multivalued mapping in G-metric space notion

We introduce some concepts on multivalued G-metric space, which will be applicable in chapter 5 to prove the coincidence fixed point results.

Kaewcharoen and Kaewkhao (2011) established the multi-valued notion in G-metric space by extending the results due to Mustafa and Sim (2006). Let X be a G-metric space. Denote by $CB(X)$ the class of all non-empty, closed and bounded subsets of X. Let $H_G(.,.,.)$ be the Haursdorff G-distance on $CB(X)$, that is, for $A, B, C \in (CB(X))$ define:

$$H_G(A,B,C) = \max\left\{ \sup_{x\in A} G(x,B,C), \sup_{x\in B} G(x,C,A), \sup_{x\in C} G(x,A,B) \right\}, \quad (2.15)$$

where,

$$G(x,B,C) = d_G(x,B) + d_G(B,C) + d_G(x,C), \quad (2.16)$$

$$d_G(x,B) = \inf\left\{ d_G(x,y), y \in B \right\}, \quad (2.17)$$

$$d_G(A,B) = \inf\left\{ d_G(a,b), a \in A, b \in B \right\}, \quad (2.18)$$

$$d_G(x,y,C) = \inf\left\{ G(x,y,z), z \in C \right\}. \quad (2.19)$$

Lemma 2.15 *(Kaewcharoen and Kaewkhao 2011) Let (X,G) be a G-metric space and $A, B \in CB(X)$. Then for each $a \in A$, we have*

$$G(a,B,B) \leq H_G((A,B,B)).$$

Tahat *et al.* (2012) gave the following lemma in G-metric space setting.

Lemma 2.16 *(Tahat* et al. *2012) Let (X,G) be a G-metric space and $A, B \in CB(X)$, then for each $\varepsilon > 0$, there exists $b \in B$ such that*

$$G(a,b,b) \leq H_G((A,B,B)) + \varepsilon.$$

2.4.4 Multivalued concept in partial metric space

We introduce a multivalued partial metric space analogue, which will help in proving our results in chapter 4, sections 4.3 and 4.4.

We obtain the description and properties of the partial Hausdorff metric from Aydi *et al.* (2012).

Let $CB^p(X)$ be a family of all non-empty, closed and bounded subsets of a partial metric space (X,p), induced by the partial metric p. Furthermore, the set A is said to be a bounded subset in (X,p) if there exists $x_0 \in X$ and $N \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,N)$. That is,

$$p(x_0,a) \leq p(a,a) + N.$$

For all $A,B \in CB^p(X)$ and $x \in X$, we define:

$$\begin{aligned} p(x,A) &= \inf\{p(x,a) : a \in A\}; \\ \delta_p(A,B) &= \sup\{p(a,B) : a \in A\}; \\ \delta_p(B,A) &= \sup\{p(b,A) : b \in B\}. \end{aligned}$$

Note that,

$$p(x,A) = 0 \implies p^s(x,A) = 0, \tag{2.20}$$

where

$$p^s(x,A) = \inf\{p^s(x,A), x \in A\}.$$

We define the partial Hausdorff metric $H_p : CB^p \times CB^p \to \mathbb{R}^+$ as

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}.$$

We state some properties of the partial Hausdorff metric H_p.

Lemma 2.17 *(Aydi et al. 2012) Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b(a) \in B$ such that*

$$p(a,b) \leq hH_p(A,B).$$

Proposition 2.4 *(Aydi et al. 2012) Let (X,p) be a partial metric space, then for any $A,B,C \in CB^p(X)$, we have*

(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$;

(ii) $\delta_p(A,A) \leq \delta_p(A,B)$;

(iii) $\delta_p(A,B) = 0 \to A \subseteq B$;

(iv) $\delta_p(A,B) = \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Proposition 2.5 *(Aydi et al. 2012) Let* (X,p) *be a partial metric space. For all* $A,B,C \in CB^p(X)$*, we have*

(H1) $H_p(A,A) \leq H_p(A,B)$;

(H2) $H_p(A,B) = H_p(B,A)$;

(H3) $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c \to C} p(c,c)$.

It is easy to see that $H_p(A,B) = 0 \Rightarrow A = B$.

Remark 2.8 *(Altun and Simsek 2012) Let* (X,p) *be a partial metric space and* A *be a nonempty subset of* X*. Then* $a \in \bar{A}$ *if and only if*

$$p(a,A) = p(a,a),$$

where $\bar{A}$ *denotes the closure of* A *with respect to the partial metric* p*. Note that* A *is closed in* (X,p) *if and only if* $\bar{A} = A$.

2.4.5 Multivalued mappings for weak partial b-metric spaces

This subsection gives some concepts on multivalued mappings for weak partial b-metric spaces, which will be used in chapter 5 to prove the main results.

Beg and Pathak (2018) proved Nadler's theorem on weak partial metric spaces with homotopy results. Later, Kanwal *et al.* (2019) defined the notion of weak partial b-metric spaces and weak partial Hausdorff b-metric space along with the topology of weak partial b-metric space. Moreover, they generalised Nadler's theorem using weak partial Hausdorff b-metric space in the context of a weak partial b-metric space. Motivated by Kanwal *et al.* (2019), we define multivalued notion in weak partial b-metric space, which is an extension of the concept given by Aydi *et al.* (2012) in partial metric space.

Let (M,ρ_b) be a weak partial b-metric space and $CB^{\rho_b}(M)$ be a class of all non-empty, closed and bounded subsets of (M,ρ_b). For $A,B \in CB^{\rho_b}(M)$ and $x \in M$, define:

$$\begin{aligned}\rho_b(x,A) &= inf\{\rho_b(x,a) : a \in A\};\\ \delta_{\rho_b}(A,B) &= \sup\{\rho_b(a,B) : a \in A\};\\ \delta_{\rho_b}(B,A) &= \sup\{\rho_b(b,A) : b \in B\}.\end{aligned}$$

Note that

$$\rho_b(x,A) = 0 \implies \rho_b^s(x,A) = 0, \tag{2.21}$$

where

$$\rho_b^s(x,A) = \inf\{\rho_b^s(x,A), x \in A\}.$$

Remark 2.9 *(Kanwal et al. 2019) Let (M,ρ_b) be a weak partial b-metric space and A a nonempty subset of M, then*

$$a \in \bar{A} \Longleftrightarrow \rho_b(a,A) = \rho_b(a,a).$$

Definition 2.33 *(Kanwal et al. 2019) Let (M,ρ_b) be a weak partial b-metric space. For $A,B \in CB^{\rho_b(M)}$, the mapping $\mathscr{H}^+_{\rho_b} : CB^{\rho_b} \times CB^{\rho_b} \to [0,\infty)$ defined by*

$$\mathscr{H}^+_{\rho_b}(A,B) = \frac{1}{2}\{\delta_{\rho_b}(A,B) + \delta_{\rho_b}(B,A)\},$$

is called $\mathscr{H}^+_{\rho_b}$-type Hausdorff metric induced by ρ_b.

2.5 Hybrid Mappings and Related Concepts

This section gives some notions on hybrid contraction mappings, which will be used in chapter 5 to prove the main results.

Hybrid fixed point theory involving pairs of single-valued and multivalued mappings is a relatively new extension in nonlinear analysis. The concepts of commutativity and weak commutativity extended to hybrid pair of mappings on metric spaces. Thus, fixed point theory has been studied, generalised and enriched in different approaches such as metric, topological and order theoretical. The method of hybrid fixed point can be applied to derive another classical fixed point theorems results and prove the existence of a bounded solution for functional equations.

Naimpally *et al.* (1986) generalised Goebel (1968) result in a hybrid of multivalued and single-valued maps satisfying a contractive condition. For the historical development of the hybrid fixed point theorem, one may refer to Mishra *et al.* (1995). Consequently, several fixed point theorems for multivalued maps were extended by Naimpally *et al.* (1986). Recently, Nashine *et al.* (2018) established a proof on common fixed point theorem for hybrid generalised (F, ψ)-contraction under common limit range property in metric spaces. Chauhan *et al.* (2014) proved unified common fixed point theorems for a hybrid pair of mappings via an implicit relation involving altering distance function.

Several researchers use the following methods to prove common fixed point theorems for hybrid pairs of mappings initiated by Sintunavarat and Kumam (2011).

Sintunavarat and Kumam (2011) introduced the notion of common limit range (CLR) property for single-valued mappings, which completely accommodated the conditions of the closeness of the degrees of the detailed mappings and showed its superiority over the (E.A) property due to Aamri and El Moutawakil (2002). Motivated by this fact, Imdad *et al.* (2014) established a common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric (semi-metric) spaces. Zoran *et al.* (2014) proved a common fixed point theorem for hybrid pair of mappings by using the notion of limit range property along with occasionally coincidentally idempotent property. Also, Imdad *et al.* (2016) established the joint common limit range notion and proved the common fixed point theorem for a pair of non-self mappings in metric space.

Some special conditions on the pairs of mappings like weakly compatible mappings, $E.A.$ property, faintly compatible mappings, common limit range property (CLR), coincidentally idempotent and joint common limit range property $(JCLR)$ has been utilised in different proofs by the researchers. One can see Popa (2016), Popa (2017). Gupta *et al.* (2020) and the references therein.

Aamri and El-moutawakil (2002) introduced the (E.A) property for pair of self mappings defined on metric spaces, which contains compatible and non-compatible mappings in metric spaces and utilised the same to prove common fixed point theorems under strict contractive conditions.

Definition 2.34 *(Aamri and El-moutawakil 2002)*
A pair of self-mappings (T,I) *of a metric space* (X,d) *is said to be non compatible if*

$\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t,$$

for some $t \in X$. But $\lim_{n\to\infty} d(TIx_n, ITx_n)$, is either non-zero or non-exist.

Definition 2.35 *(Aamri and El-moutawakil 2002)*
A pair of self-mappings (T,I) of a metric space (X,d) is said to satisfy the property (E.A) if there exist at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = t, \tag{2.22}$$

for some $t \in X$.

Imdad *et al.* (2015), extended the notion of (E.A) property to Common limit range property (CLR) and joint common limit range (JCLR) property in metric spaces for hybrid pairs of mappings as follows:

Definition 2.36 *(Imdad et al. 2015)* Let (X,d) be a metric space and $f : X \longrightarrow X$, $F : X \longrightarrow CL(X)$, (f,F) has common limit range (CLR) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Fx_n, \tag{2.23}$$

for $u \in A(X)$ and $A \in CL(X)$.

Definition 2.37 *(Imdad et al. 2015)* Let (X,d) be a metric space and $f,g : X \longrightarrow X$, $F,G : X \longrightarrow CL(X)$, the pair (f,F) and (g,G) are said to have joint common limit range (JCLR) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A,B \in CL(X)$ such that

$$\lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B, \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t, \tag{2.24}$$

such that $t \in A \cap B \in f(x) \cap g(x)$, i.e, there exist $u,v \in X$ such that $t = fu = gv \in$ A ∩ B.

2.6 Implicit relation and related concepts

We commence this section with a brief discussion on the concept of the implicit relation, which will be applicable in chapter 6 to formulate the prove of the common fixed point theorem.

Popa (1997) initiated implicit function and proved several interesting fixed point theorems satisfying suitable implicit relation. For proving such results, Popa consider τ to be the set of all continuous functions $F : \mathbb{R}^{+6} \to \mathbb{R}$ satisfying the following conditions:

(F_1) F is non-increasing in t_5 and t_6,

(F_2) there exists $h \in (0,1)$ such that for $u, v \geq 0$,

(F_{2a}) $F(u,v,v,u,u+v,0) \leq 0$,

(F_{2b}) $F(u,v,v,u,0,u+v) \leq 0 \implies u \leq hv$,

(F_3) $F(u,u,0,0,u,u) \leq 0$, for u≥ 0.

Later, Popa (1999) introduced an implicit relation to cover several well-known contractions conditions of the existing literature in one go besides admitting several new ones. Imdad *et. al.* (2002) modified results due to Popa (1999) by removing the assumption of continuity, relaxing the 'compatibility' to 'coincidentally commuting property' and replacing the completeness of the space with a set of four alternative conditions.

Recently, Ahmadullah *et al.* (2016a) proved the results for unified relation-theoretic metrical fixed point theorems of mappings satisfying implicit contractive conditions.

They considered the family $\mathscr{F}$ of all continuous real functions $F : \mathbb{R}_+^6 \longrightarrow \mathbb{R}+$ and the following conditions:

$(F1)$ F is non-increasing in the fifth variable; and $F(u,v,v,u,u+v,0) \leq 0$ for $u, v \geq 0$ implies that there exist $\lambda \in [0,1)$ such that $u \leq \lambda v$;

$(F2)$ $F(u,0,u,0,0,u) > 0$, for all $u > 0$;

$(F3)$ F is non-increasing in the sixth variable and $F(u,u,0,0,u,u) \leq 0$ for all $u > 0$.

In that way, Ahmadullah *et al.* (2016a) used this to unify and extend various findings in the literature.

We give some examples of functions that satisfy the above implicit relation conditions.

Example 2.1 The functions of $F \in \mathscr{F}$ satisfying the properties (F1) - (F3) (see, Ahmadullah *et al.* (2016a)) are the following.

(i) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - ku_2$, where $k \in [0,1)$;

(ii) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - k\{u_3 + u_4\}$, where $k \in [0,\frac{1}{2})$;

(iii) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - k\{u_5 + u_6\}$, where $k \in [0,\frac{1}{2})$;

(iv) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - a_1u_2 - a_2(u_3 + u_4) - a_3(u_5 + u_6)$, where $a_1,a_2,a_3 \in [0,1)$ and $a_1 + 2a_2 + 2a_3 < 1$;

(v) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - ku_2 - L\min\{u_3,u_4,u_5,u_6,u_6\}$, where $k \in [0,1)$ and $L \geq 0$;

(vi) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - (a_1u_2 + a_2u_3 + a_3u_4 + a_4(u_5 + u_6))$, where $a_1,a_2,a_3,a_4 \geq 0$ and $a_1 + a_2 + a_3 + a_4 < 1$;

(vii) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - k\max\left\{u_2,u_3,u_4,\dfrac{u_5 + u_6}{2}\right\} - L\min\{u_3,u_4,u_5,u_6\}$, where $k \in [0,1)$ and $L \geq 0$;

(viii) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - k\max\{u_2,u_3,u_4,u_5,u_6\}$ where $k \in [0,\frac{1}{2})$;

(ix) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - (a_1u_2 + a_2u_3 + a_3u_4 + a_4u_5 + a_5u_6))$, where $a_i^{\prime s}, \geq 0$ (for $i = 1,2,3,4,5$) and $\sum_{i=1}^{5} a_i < 1$;

(x) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - k\max\left\{u_2,u_3,u_4,\frac{u_5}{2},\frac{u_6}{2}\right\}$, where $k \in [0,1)$;

(xi) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1 - k\max\{u_2,u_3,u_4\} - (1-k)(au_5 + bu_6)$, where $k \in [0,1)$ and $0 \leqq a,b < \frac{1}{2}$.

(xii) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1^2 - u_1(a_1u_2 + a_2u_3 + a_3u_4) - a_4u_5u_6)$, where $a_1 > 0$; $a_2,a_3,a_4 \geq 0$; $a_1 + a_2 + a_3 < 1$ and $a_1 + a_4 < 1$;

(xiii) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1^2 - a_1\max\{u_2^2,u_3^2,u_4^2\} - a_2\max\{u_3u_5,u_4u_6\} - a_3u_5u_6$, where $a_i^{\prime s}, \geq 0$ (for $i = 1,2,3$); $a_1 + 2a_2 < 1$ and $a_1 + a_4 < 1$;

(xiv) $F(u_1,u_2,u_3,u_4,u_5,u_6) = u_1^3 - k\{u_2^3 + u_3^3 + u_4^3 + u_5^3 + u_6^3\}$, where $k \in [0,\frac{1}{11})$.

2.7 Relation-theoretic in metric-like spaces

This section explains relation theoretical notions in metric-like spaces concept, which will be used in chapter 6 to develop the main results.

Alam and Imdad (2015) introduced a novel variant of the Banach contraction principle on complete metric space endowed with binary relations, which under universal relation reduces to Banach contraction principle. This concept extended several results in the literature by weakening them to the extent of an arbitrary binary relation. In this context, the contraction condition is relatively weaker than the normal contraction. It must hold only on those elements related to the underlying relation rather than the whole space. Later, Ahmadullah *et al.* (2016c) extended the Banach contraction principle to metric-like spaces (not essentially complete) equipped with an arbitrary binary relation.

Next, we present some relevant relation-theoretic notions in metric-like spaces.

Definition 2.38 *(Ahmadullah et.al. 2016c)* Let (X,σ) be a metric-like space and $\mathscr{R}$ a binary relation on X. We say that (X,σ) is $\mathscr{R}$-complete if every $\mathscr{R}$-preserving Cauchy sequence $\{x_n\}$ in X, there is some $x \in X$ such that

$$\lim_{n,m\to\infty} \sigma(x_n,x_m) = \sigma(x,x) = \lim_{n\to\infty} \sigma(x_n,x).$$

Recall that the limit of a convergent sequence in metric-like spaces is not necessarily unique.

Definition 2.39 *(Ahmadullah et.al. 2016c)* Let (X,σ) be a metric-like space. Then a mapping $f : X \to X$ is said to be continuous-like at x if $fx_n \xrightarrow{\tau_\sigma} fx$ for any sequence $\{x_n\}$ with $x_n \xrightarrow{\tau_\sigma} x$. As usual, f is said to be continuous-like if it is continuous-like in the whole space X.

Definition 2.40 *(Ahmadullah et.al. 2016c)* Let (X,σ) be a metric-like space and $\mathscr{R}$ a binary relation on X. Then a mapping $f : X \to X$ is said to be $\mathscr{R}$-continuous-like at x if $fx_n \xrightarrow{\tau_\sigma} fx$ for any $\mathscr{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{\tau_\sigma} x$. As usual, f is said to be $\mathscr{R}$-continuous-like if it is $\mathscr{R}$-continuous-like in the whole space X.

Definition 2.41 *(Alam and Imdad 2015)* Let (X,σ) be a metric-like space and $\mathscr{R}$-binary relation on X. Then $\mathscr{R}$ is said to be σ-self closed if for any $\mathscr{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{\tau_\sigma} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k},x] \in \mathscr{R}$, for all $k \in \mathbb{N}$.

Definition 2.42 *(Samet and Turinici 2012)* Let (X,σ) be a metric-like space and $\mathscr{R}$-

binary relation on X. Then a subset D of X is said to be $\mathscr{R}$-directed if for every pair of point $x,y \in D$, there is z in X such that $(x,z) \in \mathscr{R}$ and $(y,z) \in \mathscr{R}$.

The following definition was given by Kolman *et al.* (2000).

Definition 2.43 *(Kolman et al. 2000)* Let (X,d) be a metric-like space, $\mathscr{R}$ a binary relation defined on X and x,y a pair of points in X. Then a finite sequence $\{z_0, z_1, z_2, ..., z_l\} \in X$ is said to be a path of length $l(where\ l\ \in N)$ joining x to y in $\mathscr{R}$ if $z_0 = x, z_l = y$ and $[z_i, z_{i+1}] \in \mathscr{R}$ for each $i \in \{1,2,3,...l-1\}$.

Observe that, a path of length l involves $(l+1)$ elements of X that need not be distinct in general. Given a metric-like space (X,σ), a self-mapping T on X and a binary relation $\mathscr{R}$ on X, we employ the following notations:

(i) $F(f)$: the set of all fixed points of f;

(ii) $X(f,\mathscr{R})$: the collection of all points $x \in X$ such that $(x,Tx) \in \mathscr{R}$;

(iii) $\gamma(x,y,\mathscr{R})$: the family of all paths joining x to y in $\mathscr{R}$.

CHAPTER THREE

COMMON FIXED POINT THEOREMS FOR F- KANNAN MAPPINGS IN GENERALISED METRIC SPACES WITH SOME APPLICATIONS

3.1 Introduction

Kannan (1968) introduced a generalisation of the Banach contraction principle. In this extension, he replaced the distance function on right-hand side in the Banach contraction inequality by the sum of the distance between the point and its image. Later, several authors used this contraction condition and generalised it to other abstract spaces.

Branciari (2000) introduced a fixed point theorem of Banach-Caccippoli type on a class of generalised metric spaces. Later, Moradi and Alimohammadi (2011) generalised Kannan's results (Kannan 1968) using the sequentially convergent property and rectangular property in metric space. Since then, several researchers have investigated Kannan's contraction mappings using rectangular property in different abstract spaces. For more details, one can see Azam and Arshad (2008), George *et al.* (2015), Malceski *et al.* (2016), Rasham *et al.* (2019a), Kari *et al.* (2020) and the references therein.

On the other hand, Beg *et al.* (2009) studied common fixed point for a pair of maps on topological vector space (TVS) valued cone metric spaces by relaxing the normality conditions imposed by Huang and Zhang (2007). They showed that the class of TVS valued cone metric spaces is larger than the class of cone metric spaces used in Radenovic and Rhoades (2009), Kadelburg *et al.* (2010), Djordjevic *et al.* (2011), Azam and Beg (2013) and the references therein. For more detail, we refer the reader to subsection 2.3.7.

This chapter uses the basic properties of generalised metric spaces defined in subsection 2.3.4 and TVS valued cone metric space defined in subsection 2.3.7 and therefore forms a basis for the remaining chapters. We denote, (X,d) as a generalised metric space and (X,ρ) as a TVS valued cone metric space. Section 3.2 deals with the basic of definitions and theorems, which are required to establish some necessary and sufficient conditions for common fixed point theorem for generalised metric spaces.

In section 3.3, we extend and generalise the results due to Batra *et al.* (2020) and Morandi and Alimohammadi (2011) using a pair of two self-mappings for F-Kannan contraction mappings in a generalised metric space, by considering the maps to be sequential convergent. In addition, some examples are provided to validate the results. In section 3.4, we investigate, extend and generalise the results due to Batra *et al.* (2020), Filipovic *et al.* (2011), Morales and Rojas (2010), Rahimi *et al.* (2013) and Wangwe and Kumar (2021) using a pair of two self-mappings in F-Kannan-Suzuki type mapping in TVS valued cone metric space, where we consider a map to be sequential, one to one and continuous by considering P to be a solid cone.

Finally, section 3.5 discusses some applications of the proved theorems, which are divided into five parts. The first part deals with the existence of a solution for the Volterra type integral equation. The second part establishes the existence of a solution for Caputo type nonlinear fractional differential equations. The third part focuses on the existence of a common solution of ordinary differential equations for damped forced oscillations. Further, the fourth part introduces the solution for the nonlinear Riemann-Liouville type fractional differential equation. Finally, the fifth part demonstrates the existence of a coincidence solution for the nonlinear Volterra-integral equation model.

We will require the following definitions and preliminary results to prove our results in this chapter.

3.2 Preliminaries

In this section, we will recall important definitions, lemmas and theorems, which will be used to prove the main results of this chapter.

Definition 3.1 *(Branciari 2000)* Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ is also convergent. Furthermore, T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

Definition 3.2 *(Munkres 1970)* Let X be a topological space. If (x_n) is a sequence of points of X, and if $n_1 < n_2 < \cdots < n_i < \ldots$ is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a subsequence of the sequence (x_n). The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

One can see more details on the sequentially convergent property (Buskes and Van Rooij 1997, Freiwald 2014).

Morandi and Alimohammadi (2011) investigated and extended the Kannan's mapping (Kannan 1968) by using the concept due to Branciari (2000), proved results on two self mappings as follows:

Theorem 3.1 *(Morandi and Alimohammadi 2011) Let (X,d) be a complete metric space and $T,S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and*

$$d(TSx, TSy) \quad \leq \quad \lambda[d(Tx, TSx) + d(Ty, TSy)], \ (x, y \in X) \tag{3.1}$$

then S has a unique fixed point. Also, if T is sequentially convergent then for every $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

The following definition is due to Beg *et al.* (2009):

Definition 3.3 *(Beg* et al. *2009) Let (X,ρ) be a topological vector space valued cone metric space, and let $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then*

(i) *$\{x_n\}_{n\geq 1}$ converges to X whenever for every $c \in E$ with $0 << c$ there is a natural number $\mathbb{N}$ such that $\rho(x_n, x) << c$ for all $n \geq \mathbb{N}$. We denote this by*

$$\lim_{n\to\infty} x_n = x \Leftrightarrow x_n \to x.$$

(ii) *$\{x_n\}_{n\geq 1}$ is Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number $\mathbb{N}$ such that $\rho(x_n, x_m) << c$ for all $n, m \geq \mathbb{N}$.*

(iii) *(X,ρ) is called topological vector space valued cone metric space if every Cauchy sequence is convergent.*

Morales and Rojas (2010) gave the following definition on cone metric space by considering P be a solid cone.

Definition 3.4 *(Morales and Rojas 2010) Let (X,d) be a cone metric space, P be a solid cone and $T : X \to X$. Then*

(i) *T is said to be continuous if*

$$\lim_{n\to\infty} x_n = x \implies \lim_{n\to\infty} Tx_n = Tx,$$

for all $x_n \in X$,

(ii) *T is said to be sequentially convergent if we have, for every sequence* (y_n)*, if* Ty_n *is convergent, then* y_n *is also convergent,*

(iii) *T is said to be subsequentially convergent, if we have for every sequence* (y_n) *and* Ty_n *is convergent implies* y_n *has a convergent subsequence.*

Filipovic *et al.* (2011) generalised Theorem 3.1 and Theorem 3.5 from (Morales and Rojas 2010) by using the sequentially convergent mappings in cone metric space and considered P to be a solid cone. They proved results on two self mappings as follows:

Definition 3.5 *(Filipovic et al. 2011) Let* (X,d) *be a cone metric space and* $T, f : X \to X$ *two mappings. A mapping* f *is said to be* T*-Hardy-Rogers contraction if there exist* $a_i \geq 0, i = 1, \ldots 5$ *with* $\sum_{i=1}^{5} a_i \leq 1$ *such that for all* $x, y \in X$,

$$\begin{aligned} d(Tfx,Tfy) \leq \; & a_1 d(Tx,Ty) + a_2 d(Tx,Tfx) + a_3 d(Ty,Tfy) \\ & + a_4 d(Tx,Tfy) + a_5 d(Ty,Tfx). \end{aligned} \tag{3.2}$$

Theorem 3.2 *(Filipovic et al. 2011) Let* (X,d) *be a complete cone metric space and* P *a solid cone, in addition let* $T : X \to X$ *be a one-to-one, continuous mapping and* $f : X \to X$ *a* T*-hardy-Rogers contraction. Then,*

(i) *For every* $x_0 \in X$*, the sequence* $Tf^n x_0$ *is Cauchy.*

(ii) *There is* $v_{x_0} \in X$ *such that* $\lim_{n\to\infty} Tf^n x_0 = v_{x_0}$.

(iii) *If* T *is sequentially convergent, then* $(f^n x_0)$ *has a convergent, subsequence.*

(iv) *There is a unique* $u_{x_0} \in X$ *such that* $fu_{x_0} = u_{x_0}$.

(v) *If* T *is sequentially convergent, then for each* $x_0 \in X$ *the iterate sequence* $(f^n x_0)$ *converges to* u_{x_0}.

Definition 3.6 *(Jeong and Rhoades 2007) Let* $F(T)$ *denote the fixed point set of a map* T*. A map* T *has property* P *if* $F(T) = F(T^n)$ *for each* $n \in \mathbb{N}$*. We shall say that a pair of maps* T *and* f *has property* Q *if* $F(T) \cap F(f) = F(T^n) \cap F(f^n)$ *for each* $n \in \mathbb{N}$.

The following theorem was given by Filipovic *et al.* (2011):

Theorem 3.3 *(Filipovic et al. 2011) Let* (X,d) *be a complete cone metric space and* P *a solid cone, in addition let* $T : X \to X$ *be a one-to-one, continuous mapping and* $f : X \to X$ *such that* $F(f) \neq \emptyset$ *and that*

$$d(Tfx, Tf^2x) \quad \preceq \quad \lambda d(Tx, Tfx), \tag{3.3}$$

holds for some $\lambda \in (0,1)$ *and for all* $x \in X, x \neq fx$*. Then* f *has property P.*

3.3 Coincidence Point Results for F-Kannan Mapping in a Generalised Metric Space

We present the main result of this section by assuming a map to be sequentially convergent with a pair of two self-mappings of F-Kannan contraction mappings. We shall start with the extension of the Definition 2.6 using a pair of two self mappings in F-Kannan mapping setting.

Definition 3.7 *Let* F *be a mapping satisfying* $(F1)-(F3)$*. A pair of two self mappings* $T,S : X \to X$ *is said to be an* F*-Kannan mapping if the following hold:*

$(FK1)$

$$TSx \neq TSy \implies TSx \neq x \text{ or } TSy \neq y. \tag{3.4}$$

$(FK2)$ *there exists* $\Upsilon > 0$ *such that*

$$\Upsilon + F(d(TSx, TSy)) \leq F\left[\frac{d(Tx, TSx) + d(Ty, TSy)}{2}\right]. \tag{3.5}$$

for all $x, y \in X$*, with* $TSx \neq TSy$*,*

By extending Remark 2.3 in Batra *et al.* (2020), we present the remark as below:

Remark 3.1 By properties of F, it follows that every F-Kannan mapping T on a metric space (X,d), satisfies the following condition:

$$d(TSx, TSy)) \leq \frac{d(Tx, TSx) + d(Ty, TSy)}{2},$$

for every $x, y \in X$.

We give the following examples from Batra *et al.* (2020) in the context of a pair of two mappings:

Example 3.12 Let $F_1 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F_1(z) = \ln(z)$. Then clearly, $(F1)-(F3)$ are satisfied by $F_1(z)$. In fact, $(F3)$ holds for every $k \in (0,1)$. Moreover, condition (3.5) above takes the form:

$$d(TSx,TSy)) \leq e^{-\Upsilon}\left[\frac{d(Tx,TSx)+d(Ty,TSy)}{2}\right]. \tag{3.6}$$

for all $x,y \in X$ with $TSx \neq TSy$.

Thus, if $T,S : X \to X$ is a Kannan mapping with constant $k \in (0,1)$ satisfying

$$d(TSx,TSy)) \leq k\left[\frac{d(Tx,TSx)+d(Ty,TSy)}{2}\right]. \tag{3.7}$$

for every $x,y \in X$. Then it also satisfies (3.6) and (3.5) with $\Upsilon = ln\frac{1}{k}$.
Example 3.13 Let $F_2 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F_2(z) = \ln(z) + z, z > 0$. Then, $(F1)-(F3)$ are satisfied by $F_2(z)$. Condition (3.5) above takes the form:

$$\frac{d(TSx,TSy)}{(d(Tx,TSx)+d(Ty,TSy))/2}e^{d(TSx,TSy)-\frac{d(Tx,TSx)+d(Ty,TSy)}{2}} \leq e^{-\Upsilon}. \tag{3.8}$$

for all $x,y \in X$ with $TSx \neq TSy$.

We proof the following lemma by extending lemma 2.2, which will be useful in the prove of the main theorem.

Lemma 3.1 *Let (X,d) be a metric space and $T,S : X \to X$ be an F-Kannan mapping. Then*

$$d(TS^i x_0, TS^{i+1}x_0) \to 0 \text{ as } i \to \infty$$

for all $x \in X$.

Proof. Let $x_0 \in X$ be arbitrary. If $TS^i x_0 = TS^{i+1}x_0$ for some $i \in \mathbb{N}$, then the sequence $\{x_i\}_{n\in\mathbb{N}}$ converges in X and hence the sequence $d(TS^i x_0, TS^{i+1}x_0) \to 0$ *as* $i \to \infty$ for all $x \in X$.

Assume that $TS^i x_0 \neq TS^{i+1}x_0$ for any $i \in \mathbb{N}$. Then, by (3.5) with $\Upsilon > 0$ we get

$$\Upsilon + F(d(TS^i x_0, TS^{i+1}x_0)) \leq F\left[\frac{d(TS^{i-1}x_0, TS^i x_0) + d(TS^i x_0, TS^{i+1}x_0)}{2}\right]. \tag{3.9}$$

From Remark 3.1 we have

$$d(TS^i x_0, TS^{i+1}x_0) \leq \frac{d(TS^{i-1}x_0, TS^i x_0) + d(TS^i x_0, TS^{i+1}x_0)}{2}. \tag{3.10}$$

Using (3.10) in (3.9), as results yields to

$$\Upsilon + F(d(TS^i x_0, TS^{i+1}x_0)) \leq F(d(TS^i x_0, TS^{i+1}x_0)). \tag{3.11}$$

Letting $i \to \infty$ in (3.11), we get

$$\begin{aligned} \Upsilon + 0 &\leq 0, \\ \Upsilon &\leq 0, \end{aligned}$$

which is a contradiction. Hence $d(TS^i x_0, TS^{i+1}x_0) \to 0$ as $i \to \infty$. □

Now we give an existence results based on sequentially convergent property. Motivated by Morandi and Alimohammadi (2011) and Batra *et al.* (2020), we prove the extended version of Theorem 3.1 and Theorem 2.9 using F-Kannan mappings with a pair of two self mappings in generalised metric space.

Theorem 3.4 *Let (X,d) be a complete generalised metric space and $T,S : X \to X$ be an F-Kannan mapping such that T is continuous, injection and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$, $\Upsilon > 0$ and*

$$\Upsilon + F(d(TSx, TSy)) \leq F\left[\frac{d(Tx,TSx) + d(Ty,TSy)}{2}\right], \tag{3.12}$$

then S has a unique fixed point. Also if T is subsequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^i x_0\}$ converges to this fixed point.

Proof. Assume $x_0 \in X$ be an arbitrary point in X. Let the sequence $\{x_i\}_{i\geq 1}$ be defined by $x_{i+1} = Sx_i$ and $x_i = S^i x_0$, for $i = 1, 2, \ldots$

Using inequality (3.5), we obtain

$$\begin{aligned} d(Tx_i, Tx_{i+1}) &= d(TSx_{i-1}, TSx_i). \\ \Upsilon + F(d(Tx_i, Tx_{i+1})) &\leq F\left[\frac{d(Tx_{i-1}, TSx_{i-1}) + d(Tx_i, TSx_i)}{2}\right], \\ F(d(Tx_i, Tx_{i+1})) &\leq F\left[\frac{d(Tx_{i-1}, TSx_{i-1}) + d(Tx_i, TSx_i)}{2}\right] - \Upsilon. \end{aligned}$$

Since F is strictly increasing, we deduce

$$\begin{aligned} d(Tx_i, Tx_{i+1}) &< \frac{d(Tx_{i-1}, TSx_{i-1}) + d(Tx_i, TSx_i)}{2}, \\ d(Tx_i, Tx_{i+1}) &< \frac{d(Tx_{i-1}, Tx_i) + d(Tx_i, Tx_{i+1})}{2}, \end{aligned}$$

and hence

$$\begin{aligned} 2d(Tx_i,Tx_{i+1})-d(Tx_i,Tx_{i+1}) &< d(Tx_{i-1},Tx_i), \\ d(Tx_i,Tx_{i+1}) &< d(Tx_{i-1},Tx_i), \end{aligned}$$

By $(F1)$, this implies that

$$F(d(Tx_i,Tx_{i+1})) \;<\; F(d(Tx_{i-1},Tx_i)).$$

Consequently, we get

$$\Upsilon + F(d(Tx_i,Tx_{i+1})) \;\leq\; F(d(Tx_{i-1},Tx_i))$$

so,

$$F(d(Tx_i,Tx_{i+1})) \;\leq\; F\Big[d(Tx_{i-1},Tx_i)\Big]-\Upsilon. \tag{3.13}$$

Similarly, we obtain

$$d(Tx_{i+1},Tx_{i+2}) = d(TSx_i,TSx_{i+1})$$

$$F(d(Tx_{i+1},Tx_{i+2})) \;\leq\; F\left[\frac{d(Tx_i,TSx_i)+d(Tx_{i+1},TSx_{i+1})}{2}\right]-\Upsilon.$$

Which implies that

$$F(d(Tx_{i+1},Tx_{i+2})) \;\leq\; F[d(Tx_i,Tx_{i+1})]-\Upsilon. \tag{3.14}$$

Applying (3.13) in (3.14) we obtain

$$F(d(Tx_{i+1},Tx_{i+2})) \;\leq\; F[d(Tx_{i-1},Tx_i)]-2\Upsilon, \tag{3.15}$$

by $(F1)$.

Using induction and (3.13) we deduce

$$F(d(Tx_i,Tx_{i+1})) \;\leq\; F[d(Tx_{i-1},Tx_i)]-i\Upsilon. \tag{3.16}$$

Letting $i \to \infty$ in (3.16) and using condition $(F2)$ of F results in

$$\lim_{n\to\infty} d(Tx_i,Tx_{i+1}) \;=\; 0.$$

By Lemma 3.1, we have $d(Tx_i,Tx_{i+1}) \to 0$ as $i \to \infty$. Denote $d(Tx_i,Tx_{i+1}) = \alpha_i$, for all $i = 1,2,3,\dots$ and $i \in \mathbb{N}$, for F-Kannan mappings.

Using condition $(F3)$ of the function F there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (\alpha_i)^k F(\alpha_i) = 0. \tag{3.17}$$

From (3.16), for every $i \in \mathbb{N}$, we have

$$\begin{aligned} (\alpha_i)^k F(\alpha_i) &\leq \cdots \leq (\alpha_n)^k F(\alpha_{i-1}) - i\Upsilon(\alpha_i)^k, \\ (\alpha_i)^k F(\alpha_i) - (\alpha_i)^k F(\alpha_{i-1}) &\leq -i\Upsilon(\alpha_i)^k, \\ (\alpha_i)^k [F(\alpha_i) - F(\alpha_{i-1})] &\leq -i\Upsilon(\alpha_i)^k \leq 0. \end{aligned} \tag{3.18}$$

On taking limit as $i \to \infty$ in (3.18), we get

$$\lim_{i \to \infty} i(\alpha_i)^k = 0. \tag{3.19}$$

From (3.19), there exist $i_1 \in \mathbb{N}$ such that $i(\alpha_i)^k \leq 1$, for all $i \geq i_1$, which follows that

$$\alpha_i \leq i^{-\frac{1}{k}}, \forall i \geq i_1. \tag{3.20}$$

Therefore, $\sum_{i=0}^{\infty} d(Tx_i, Tx_{i+1})$ converges.

By (3.20), we prove that $\{Tx_i\}$ is a Cauchy sequence since (X,d) is complete. Consider $i, j \in N$ such that $j \geq i$,

$$\begin{aligned} d(Tx_i, Tx_j) &\leq d(Tx_i, Tx_{i+1}) + d(Tx_{i+1}, Tx_{i+2}) + d(Tx_{i+2}, Tx_{i+3}) + \ldots \\ &\quad + d(Tx_{j-1}, Tx_j), \\ &\leq \alpha_i + \alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_{j-1}, \\ &= \sum_{i}^{j-1} d(Tx_i, Tx_{i+1}) \\ &\leq \sum_{j=i}^{\infty} \alpha_i \\ &\leq \sum_{i=1}^{\infty} i^{-\frac{1}{k}}. \end{aligned}$$

This shows that the series $\sum_{i=1}^{\infty} i^{-\frac{1}{k}}$ converges, which implies that

$$\lim_{i \to \infty} d(Tx_i, Tx_j) = 0.$$

So, $Tx_i = Tx_j$ for every $j \geq i$ in X. Hence $\{Tx_i\}$ is a Cauchy sequence in X. The completeness of X ensures the existence of $x^* \in X$ such that

$$\begin{aligned} d(Tx^*, x^*) &= \lim_{i,j \to \infty} d(Tx_i, Tx_j) = 0, \\ &= \lim_{i \to \infty} d(Tx_i, x^*) = 0. \end{aligned} \tag{3.21}$$

By (3.21) it follows that $x_{i+1} \to x^*$ as $i \longrightarrow \infty$. By continuity of S and T, we have

$$\begin{aligned} x^* &= \lim_{i \to \infty} x_i = \lim_{i \to \infty} x_{i+1} = \lim_{i \to \infty} Sx_i = Sx^*. \\ x^* &= \lim_{i \to \infty} x_i = \lim_{i \to \infty} x_{i+1} = \lim_{i \to \infty} Tx_i = Tx^*. \end{aligned}$$

Since X is a complete metric space, there exists $x^* \in X$ such that

$$\lim_{i\to\infty} Tx_i \quad = \quad x^*. \tag{3.22}$$

Now, we prove that x^* is a fixed point of T. Thus, by (iii) of Definition 3.3 we have

$$d(x^*, Tx^*) \quad \leq \quad d(x^*, Tx_i) + d(Tx_i, Tx_{i+1}) + d(Tx_{i+1}, Tx^*), \tag{3.23}$$

By Remark 3.1, it implies that

$$d(Tx_{i+1}, Tx^*) \quad \leq \quad \frac{d(Tx_i, Tx_{i+1}) + d(Tx_{i+1}, Tx^*)}{2}. \tag{3.24}$$

Applying (3.24) in (3.23), we obtain

$$d(x^*, Tx^*) \quad \leq \quad d(x^*, x^*) + d(x^*, x^*) + \frac{d(x^*, Tx^*) + d(x^*, Tx^*)}{2}.$$

Letting $i \to \infty$ and using Lemma 3.1 in above inequality, we get

$$\begin{aligned}
d(x^*, Tx^*) &\leq d(x^*, Tx_i) + d(Tx_i, Tx_{i+1}) + \frac{d(Tx_i, Tx_{i+1}) + d(Tx_{i+1}, Tx^*)}{2}, \\
&\leq d(x^*, x^*) + d(x^*, x^*) + \frac{d(x^*, x^*) + d(x^*, Tx^*)}{2}, \\
d(x^*, Tx^*) &\leq \frac{d(x^*, Tx^*)}{2}, \\
2d(x^*, Tx^*) &\leq d(x^*, Tx^*), \\
2d(x^*, Tx^*) &\leq d(x^*, Tx^*), \\
d(x^*, Tx^*) &\leq 0.
\end{aligned}$$

That is, $Tx^* = x^*$.

Next, we prove that x^* is a unique fixed point of T. Assume the contrary, i.e, there exists $w^* \in Card\{Fix\ T\}$ such that $x^* \neq w^*$. Let $Tx_i \to w^*$ and w^* is a fixed point of T. Using Remark 3.1 and Lemma 3.1, it follows that $x^* = w^*$ which is a contradiction. Thus, T is a PO on X.

Moreover, T is a subsequentially convergent, $\{x_i\}$ has a convergent subsequence, there exists $w^* \in X$ and $\{x_{i(k)}\}_{k=1}^{\infty}$ so that $\lim_{k\to\infty} x_{i(k)} = w^*$. Since T is continuous and

$$\lim_{k\to\infty} x_{i(k)} = w^*.$$

Due to the continuity of T, it implies that

$$\lim_{k\to\infty} Tx_{i(k)} \quad = \quad Tw^*.$$

By (3.22), we conclude that

$$Tw^* \quad = \quad x^*.$$

Using Remark 3.1 and (ii) of Definition 3.3, we get

$$\begin{aligned} d(Tx_i,Tx_{i+1}) &= d(TSx_{i-1},TSx_i), \\ &\leq \lambda(d(Tx_{i-1},TSx_{i-1})+d(Tx_i,TSx_i)), \\ &= \lambda(d(Tx_{i-1},TSx_{i-1})+d(Tx_i,Tx_{i+1})), \\ &\leq \frac{\lambda}{1-\lambda}d(Tx_{i-1},TSx_{i-1}). \end{aligned} \tag{3.25}$$

Thus, using Equation 2.9 and (iii) of Definition 2.12 we have

$$\begin{aligned} \Upsilon+F(d(Tx_{i+1},Tx_{i+2})) &= \Upsilon+F(d(TSx_i,TSx_{i+1})). \\ F(d(TSw^*,Tw^*)) &\leq F(d(TSw^*,Tx_i)+d(Tx_i,Tx_{i+1})+d(Tx_{i+1}),Tw^*)). \end{aligned}$$

As F is sequentially increasing, this implies that

$$\begin{aligned} d(TSw^*,Tw^*) &< d(TSw^*,Tx_i)+d(Tx_i,Tx_{i+1})+d(Tx_{i+1},Tw^*), \\ d(TSw^*,Tw^*) &\leq d(TSw^*,TS^{i(k)}x_0)+d(TS^{i(k)}x_0,TS^{i(k)+1}x_0) \\ &\quad +d(TS^{i(k)+1}x_0,Tw^*). \end{aligned} \tag{3.26}$$

By Lemma 3.1 when $TS^ix_0 \neq TS^{i+1}x_0$ for any $i \in \mathbb{N}$ and (3.25), we obtain

$$\begin{aligned} d(TSw^*,TS^{i(k)}x_0) \qquad &\leq \lambda[d(Tw^*,TSw^*)+d(TS^{i(k)-1}x_0,TS^{i(k)}x_0)] \\ &\leq \lambda d(Tw^*,TSw^*)+\lambda\left(\frac{\lambda}{1-\lambda}\right)^{i(k)-1}d(Tx_0,Tx_1) \end{aligned} \tag{3.27}$$

$$d(TS^{i(k)+1}x_0,TS^{i(k)}x_0) \leq \left(\frac{\lambda}{1-\lambda}\right)^{i(k)}d(Tx_1,Tx_0). \tag{3.28}$$

Using (3.27) and (3.28) in (3.26), we obtain

$$\begin{aligned} d(TSw^*,Tw^*) &\leq \lambda d(Tw^*,TSw^*)+\lambda\left(\frac{\lambda}{1-\lambda}\right)^{i(k)-1}d(Tx_0,Tx_1) \\ &\quad +\left(\frac{\lambda}{1-\lambda}\right)^{i(k)}d(Tx_1,Tx_0)+d(Tx_{i(k)+1},Tw^*). \end{aligned}$$

Which follows

$$\begin{aligned} d(TSw^*,Tw^*) &\leq \left(\frac{\lambda}{1-\lambda}\right)^{i(k)}d(Tx_0,Tx_1)+\left(\frac{\lambda}{1-\lambda}\right)^{i(k)+1}d(Tx_1,Tx_0) \\ &\quad +\left(\frac{1}{1-\lambda}\right)d(Tx_{i(k)+1},Tw^*). \end{aligned} \tag{3.29}$$

Letting $k \to \infty$ in (3.29), we obtain

$$d(TSw^*,Tw^*) = 0.$$

Since T is injection, $Sw^* = w^*$. So, S has a fixed point. As T is sequentially convergent, we conclude that $\{x_i\}$ converges to the fixed point of S. Implies that $Sx \subset X$ and $Tx \subset X$, then, there exists a point $w^* \subset X$ such that $w^* \in Sw^* \cap Tw^*$, that is, w^* is a common fixed point of S and T. which satisfies all fundamental properties of Definition 2.3. $\square$

Dasgupta *et al.* (2014), and Moradi and Alimohammadi (2011), in their work, considered an example in which Kannan Theorem is not applicable. At the same time, generalised Kannan mappings imply the existence of a fixed point for the considered mapping. We use one more example of this type in this work, which satisfies F-Kannan mapping.

Example 3.14 Consider the sequence $X = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ and d be an Euclidean metric on X. Then (X,d) is a complete metric space. Let the mapping $S : X \to X$ be determined as follows:

$$\begin{aligned} S(0) &= 0 \\ S(1/n) &= \frac{1}{n+1}. \end{aligned}$$

for $n \geq 1$. Let there exists $\lambda \in [0, \frac{1}{2})$, so that for all $x, y \in X$ condition (2.1) is satisfied although is not true for every $\lambda > 0$. Which is a contradiction, hence Kannan's theorem can not be applicable.

The mapping $T : X \to X$ be determined as

$$\begin{aligned} T(0) &= 0 \\ T(1/n) &= \frac{1}{n^n}. \end{aligned}$$

For all $n > 1$, T is continuous, injection, and subsequentially convergent.

Now, let $m, n \in \mathbb{N}$, $m > n$. Then, we prove that (T,S) is an F-Kannan mappings with respect to $F_2(z) = \ln z + z$ and $\Upsilon = 1$.

By using $(KF2)$ with $F_2(z)$, we note that (3.12) becomes

$$\frac{d(TSx,TSy)}{[d(Tx,TSx)+d(Ty,TSy)]/2} e^{d(TSx,TSy) - [\frac{d(Tx,TSx)+d(Ty,TSy)}{2}]} \leq e^{-\Upsilon}. \tag{3.30}$$

To see this, we now calculate $d(Sx,Ty)$ for $x = \frac{1}{m}$, $y = \frac{1}{n}$, $n \geq 1$.

$$\begin{aligned} d(TSx,TSy) &= d(TS(1/m), TS(1/n)), \\ &\leq \left| \frac{1}{(n+1)^{n+1}} - \frac{1}{(m+1)^{m+1}} \right|. && (3.31) \\ d(Tx,TSx) &= d(T(\frac{1}{m}), TS(\frac{1}{m})), \\ &\leq \frac{1}{3} \left| \frac{1}{m} - \frac{1}{(m+1)^{m+1}} \right|. && (3.32) \\ d(Ty,TSy) &= d(T(\frac{1}{n}), TS(\frac{1}{n})), \\ &\leq \frac{1}{3} \left| \frac{1}{n} - \frac{1}{(n+1)^{n+1}} \right|. && (3.33) \end{aligned}$$

Applying $(3.31),(3.32)$ and (3.33) in (3.30) becomes

$$\frac{|TS(1/m),TS(1/n)|}{[|T(\frac{1}{m}),TS(\frac{1}{m})|+|T(\frac{1}{n}),TS(\frac{1}{n})|]/6}e^{|TS(1/m),TS(1/n)|-[\frac{|T(\frac{1}{m}),TS(\frac{1}{m})|+|T(\frac{1}{n}),TS(\frac{1}{n})|}{6}]} \leq e^{-\Upsilon}.$$

By Theorem 3.4, S has unique fixed point that is $x^* = 0$.

3.4 Existence Results for F-Kannan-Suzuki Type Mappings in TVS Valued Cone Metric Spaces

Now, we obtain a more general version of Definitions 2.5 and 2.6 using a pair of two self mappings in F-Kannan-Suzuki type mapping setting. Let (X,ρ) denote a TVS valued cone metric space.

Definition 3.8 *Let F be a mapping satisfying $(F1)-(F3)$. A pair of two self mappings $G,f : X \to X$ is said to be an F-Kannan-Suzuki type mapping if the following hold:*

$(FKS1)$

$$Gfx \neq Gfy \implies Gfx \neq x \text{ or } Gfy \neq y. \tag{3.34}$$

$(FKS2)$ *there exists $\vartheta > 0$ such that*

$$\begin{aligned} \frac{1}{2}\rho(x,Gx) &< \rho(x,y) \\ \implies \vartheta + F(\rho(Gfx,Gfy)) &\leq F\left[\frac{\rho(Gx,Gfx)+\rho(Gy,Gfy)}{2}\right], \end{aligned} \tag{3.35}$$

for all $x,y \in X$, with $Gfx \neq Gfy$ and $F \in \mathfrak{F}$.

The following remark is motivated by the work of Batra *et al.* (2020) given below:

Remark 3.2 *By properties of F, it follows that every F-Kannan-Suzuki type mapping G on a TVS-valued cone metric space (X,ρ), satisfies following condition:*

$$\rho(Gfx,Gfy)) \leq \frac{\rho(Gx,Gfx)+\rho(Gy,Gfy)}{2},$$

for every $x,y \in X$.

We give the following examples in the context of a pair of two mappings:

Example 3.15 *Let $F_1 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F_1(z) = \ln(z)$. Then clearly, $(F1),(F2'')$ and $(F3'')$ are satisfied by $F_1(z)$. Moreover, condition (3.35) above takes the form:*

$$\rho(Gfx,Gfy)) \leq e^{-\vartheta}\left[\frac{\rho(Gx,Gfx)+\rho(Gy,Gfy)}{2}\right]. \tag{3.36}$$

for all $x,y \in X$ with $Gfx \neq Gfy$.

Thus, if $G,f : X \to X$ is a Kannan mapping with constant $k \in (0,1)$ satisfying

$$\rho(Gfx,Gfy)) \leq k\left[\frac{\rho(Gx,Gfx)+\rho(Gy,Gfy)}{2}\right]. \tag{3.37}$$

for every $x,y \in X$. Then it also satisfies (3.36) and (3.37) with $\vartheta = ln\frac{1}{k}$.

Example 3.16 *Let $F_2 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F_2(z) = -\frac{1}{z}, z > 0$. Then, $(F1),(F2'')$ and $(F3'')$ are satisfied by $F_2(z)$. Condition (3.35) above takes the form:*

$$\frac{\rho(Gx,Gfx)+\rho(Gy,Gfy)}{2} \leq \frac{\rho(Gfx,Gfy)}{1-\vartheta\rho(Gfx,Gfy)}. \tag{3.38}$$

for all $x,y \in X$ with $Gfx \neq Gfy$.

Example 3.17 *Let $F_3 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $F_3(z) = -\frac{1}{z} + z, z > 0$. Then, $(F1),(F2'')$ and $(F3'')$ are satisfied by $F_3(z)$. Condition (3.35) above takes the form:*

$$\frac{\rho(Gx,Gfx)+\rho(Gy,Gfy)}{2} \leq \frac{\rho(Gfx,Gfy)\Big(\Big[\frac{\rho(Gx,Gfx)+\rho(Gy,Gfy)}{2}\Big]^2-1\Big)}{\rho(Gfx,Gfy)+\vartheta(\rho(Gfx,Gfy)^2-1)}. \tag{3.39}$$

for all $x,y \in X$ with $Gfx \neq Gfy$.

Motivated by Filipovic *et al.* (2011) and Batra *et al.* (2020), we prove the extended version of Theorem 2.8, 2.9 and 3.2 using F-Kannan-Suzuki type mappings with a pair of two self mappings in TVS-valued cone metric space.

Theorem 3.5 *Let (X,ρ) be a complete TVS-valued cone metric space and P a solid cone, in addition let $G : X \to X$ be a one-to-one, continuous mapping and $f : X \to X$ a G-F-Kannan-Suzuki type contraction. Then*

- (i) *For every $x_0 \in X$, the sequence $Gf^n x_0$ is convergent.*
- (ii) *If G is sequentially convergent, then for each $x_0 \in X$ the iterate sequence $(f^n x_0)$ converges to u^*.*
- (iii) *There is a unique $u^* \in X$ such that $fu^* = u^*$.*
- (iv) *There is $v^* \in X$ such that $\lim_{n\to\infty} Gf^n x_0 = v^*$.*
- (v) *If G is sequentially convergent, then $(f^n x_0)$ has a convergent subsequence.*

Proof. We start by constructing a Cauchy sequence using (i) as follows: Let $x_0 \in X$ be arbitrary. If $Gf^n x_0 = Gf^{n+1} x_0$ for some $n \in \mathbb{N}$, then sequence $\{x_n\}_{n\in\mathbb{N}}$ converges in X

and hence the sequence $\rho(Gf^nx_0, Gf^{n+1}x_0) \to 0 \; as \; n \to \infty$ for all $x \in X$. Suppose that $Gf^nx_0 \neq Gf^{n+1}x_0$ for any $n \in \mathbb{N}$. Then, by (3.35) with $\vartheta > 0$ we get

$$\begin{aligned} \frac{1}{2}\rho(x_n, Gx_n) &< \rho(x_n, Gx_n) \Longrightarrow \\ \vartheta + F(\rho(Gf^nx_0, Gf^{n+1}x_0)) &\leq F\left[\frac{\rho(Gf^{n-1}x_0, Gf^nx_0) + \rho(Gf^nx_0, Gf^{n+1}x_0)}{2}\right]. \end{aligned} \tag{3.40}$$

From Remark 3.2 we have

$$\begin{aligned} \rho(Gf^nx_0, Gf^{n+1}x_0) &\leq \frac{\rho(Gf^{n-1}x_0, Gf^nx_0) + \rho(Gf^nx_0, Gf^{n+1}x_0)}{2}, \\ 2\rho(Gf^nx_0, Gf^{n+1}x_0) &\leq \rho(Gf^{n-1}x_0, Gf^nx_0) + \rho(Gf^nx_0, Gf^{n+1}x_0), \\ \rho(Gf^nx_0, Gf^{n+1}x_0) &\leq \rho(Gf^{n-1}x_0, Gf^nx_0). \end{aligned} \tag{3.41}$$

Using (3.40) in (3.41), we get

$$\begin{aligned} \vartheta + F(\rho(Gf^nx_0, Gf^{n+1}x_0)) &\leq F\left[\frac{\rho(Gf^{n-1}x_0, Gf^nx_0) + \rho(Gf^nx_0, Gf^{n+1}x_0)}{2}\right], \\ \vartheta + F(\rho(Gf^nx_0, Gf^{n+1}x_0)) &\leq F\left[\frac{2\rho(Gf^{n-1}x_0, Gf^nx_0)}{2}\right], \\ \vartheta + F(\rho(Gf^nx, Gf^{n+1}x)) &\leq F\Big[\rho(Gf^{n-1}x, Gf^nx)\Big]. \end{aligned} \tag{3.42}$$

Letting $n \to \infty$ in (3.42), we get

$$\begin{aligned} \vartheta + 0 &\leq 0, \\ \vartheta &\leq 0, \end{aligned}$$

Since $\vartheta > 0$ is a contradiction. Hence $\rho(Gf^nx_0, Gf^{n+1}x_0) \to 0$ as $n \to \infty$. Thus $\{Gf^nx_0\}$ converges.

Since G is sequentially convergent, using condition (ii) we prove that the iterate of a sequence f^nx_0 converge to a fixed point u^* in X. To see this, suppose $x_0 \in X$ be an arbitrary point in X. Let the sequence $\{x_n\}_{n\geq 1}$ be defined by $x_{n+1} = fx_n = f^{n+1}x_0 = ff^nx_0$ and $x_n = fx_{n-1} = f^nx_0 = ff^{n-1}x_0$, for $n \geq 1 \in \mathbb{N}$. Thus , we have

$$\rho(x_n, x_{n+1}) \leq \rho(fx_{n-1}, fx_n) = \rho(f^nx_0, f^{n+1}x_0) = \rho(ff^{n-1}x_0, ff^nx_0).$$

Equivalently to

$$\begin{aligned} \rho(Gx_n, Gx_{n+1}) &\leq \rho(Gfx_{n-1}, Gfx_n) = \rho(Gf^nx_0, Gf^{n+1}x_0) \\ &= \rho(Gff^{n-1}x_0, Gff^nx_0). \end{aligned}$$

Let $x = f^{n-1}x_0$ and $y = f^n x_0$, using inequality (3.35), we obtain

$$\begin{aligned}
\frac{1}{2}\rho(x_n, Gx_n) &< \rho(x_n, Gx_n) \Longrightarrow \\
\vartheta + F(\rho(Gff^{n-1}x_0, Gff^n x_0) &\leq F\left[\frac{\rho(Gf^{n-1}x_0, Gff^{n-1}x_0) + \rho(Gf^n x_0, Gff^n x_0)}{2}\right], \\
F(\rho(Gx_n, Gx_{n+1})) &\leq F\left[\frac{\rho(Gx_{n-1}, Gx_n) + \rho(Gx_n, Gx_{n+1})}{2}\right] - \vartheta.
\end{aligned}$$

Since F is strictly increasing, by using Remark 3.2, we deduce

$$\begin{aligned}
\rho(Gff^{n-1}x_0, Gff^n x_0) &\leq \frac{\rho(Gf^{n-1}x_0, Gff^{n-1}x_0) + \rho(Gf^n x_0, Gff^n x_0)}{2}, \\
\rho(Gx_n, Gx_{n+1}) &< \frac{\rho(Gx_{n-1}, Gx_n) + \rho(Gx_n, Gx_{n+1})}{2},
\end{aligned}$$

and hence

$$\begin{aligned}
2\rho(Gx_n, Gx_{n+1}) - \rho(Gx_n, Gx_{n+1}) &< \rho(Gx_{n-1}, Gx_n), \\
\rho(Gx_n, Gx_{n+1}) &< \rho(Gx_{n-1}, Gx_n),
\end{aligned}$$

By $(F1)$, this implies that

$$F(\rho(Gx_n, Gx_{n+1})) < F(\rho(Gx_{n-1}, Gx_n)).$$

Consequently, we get

$$\vartheta + F(\rho(Gx_n, Gx_{n+1})) \leq F(\rho(Gx_{n-1}, Gx_n))$$

so,

$$F(\rho(Gx_n, Gx_{n+1})) \leq F(\rho(Gx_{n-1}, Gx_n)) - \vartheta. \tag{3.43}$$

By induction and (3.43) we deduce

$$\begin{aligned}
F(\rho(Gx_{n+1}, Gx_{n+2})) &\leq F(\rho(Gx_{n-1}, Gx_n)) - 2\vartheta. \\
F(\rho(Gx_{n+2}, Gx_{n+3})) &\leq F(\rho(Gx_{n-1}, Gx_n)) - 3\vartheta. \\
\Longrightarrow F(\rho(Gx_n, Gx_{n+1})) &\leq F(\rho(Gx_{n-1}, Gx_n)) - n\vartheta. \qquad (3.44)
\end{aligned}$$

Letting $n \to \infty$ in (3.44), we find that

$$\lim_{n\to\infty} F(\rho(Gx_n, Gx_{n+1})) = -\infty.$$

Cosequently, using Lemma 2.1 and condition ($F2$”) of F results in

$$\lim_{n\to\infty} \rho(Gx_n, Gx_{n+1}) = 0. \tag{3.45}$$

Thus, there exists $n \in \mathbb{N}$ such that

$$\rho(Gx_n, Gx_{n+1}) \;<\; \rho(Gx_n, G^2x_n) < c\rho(x_n, Gx_n) < \rho(x_n, Gx_n),$$

which is a contradiction. Hence, we have

$$\lim_{n\to\infty} \rho(x_n, Gx_n) \;=\; 0.$$

Therefore, we have $\rho(Gx_n, Gx_{n+1}) \to 0$ as $n \to \infty$. Denote $\alpha_n = \rho(Gx_n, Gx_{n+1}) = 0$, for all $n \geq 1$ and $n \in \mathbb{N}$, for F-Kannan-Suzuki type mappings.

By (3.45), we prove that $\{Gx_n\}$ is a Cauchy sequence since (X, ρ) is complete. Consider $n, m \in N$ such that $m > n$. Assume on the contrary, that there exists $c > 0$ and sequences $\{p(n)\}_{n\geq 1}^{\infty}$ and $\{q(n)\}_{n\geq 1}^{\infty}$ such that

$$p(n) > q(n) > n,\; \rho(Gx_{p(n)}, Gx_{q(n)}) \geq c,\; \rho(Gx_{p(n)-1}, Gx_{q(n)}) \leq c, \forall n \in \mathbb{N}.$$

Using (iii) of Definition 2.18, we get

$$\begin{aligned} \rho(Gx_{p(n)}, Gx_{q(n)}) &\leq \rho(Gx_{p(n)}, Gx_{p(n)-1}) + \rho(Gx_{p(n)-1}, Gx_{q(n)}), \\ &\leq \rho(Gx_{p(n)}, Gx_{p(n)-1}) + c. \end{aligned}$$

It follows from (3.45) and above inequality

$$\lim_{n\to\infty} \rho(Gx_{p(n)}, Gx_{q(n)}) = c. \tag{3.46}$$

From $(F3")$, (3.46) and (3.35), we get

$$\vartheta + F(\rho(Gx_{p(n)}, Gx_{q(n)})) \;\leq\; F\left[\frac{\rho(Gx_{p(n)-1}, Gx_{p(n)}) + \rho(Gx_{p(n)}, Gx_{q(n)})}{2}\right].$$

Equivalently to

$$\begin{aligned} \vartheta + F(c) &\leq F(c), \\ \vartheta &\leq 0, \end{aligned}$$

which is a contradiction. So, $Gx_n = Gx_m$ for every $m \geq n$ in X. Hence $\{Gx_n\}$ is a Cauchy sequence in X. The completeness of X ensures the existence of $u^* \in X$ such that

$$\begin{aligned} d(Gu^*, u^*) &= \lim_{n,m\to\infty} \rho(Gx_n, Gx_m) = 0, \\ &= \lim_{n\to\infty} \rho(Gx_n, u^*) = 0. \end{aligned} \tag{3.47}$$

From (3.47), (iii) and Definition 3.4 it follows that $Gx_{n+1} \to u^*$ as $n \longrightarrow \infty$. By sequential continuity of f and G, we have

$$\begin{aligned} u^* &= \lim_{n\to\infty} f^n x_0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu^*. \\ u^* &= \lim_{n\to\infty} Gf^n x_0 = \lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Gx_{n+1} = \lim_{u\to\infty} G^2x_n = Gu^*. \end{aligned}$$

Since X is a complete metric space, there exists $u^* \in X$ such that

$$\lim_{n\to\infty} Gx_n = Gu^* = u^*. \tag{3.48}$$

Now, we prove that u^* is a fixed point of G. Thus, by (iii) of Definition 2.18 and $\rho(u^*, Gu^*) \geq 0$, we have

$$\rho(u^*, Gu^*) \leq \rho(u^*, Gx_{n+1}) + \rho(Gx_{n+1}, Gu^*), \tag{3.49}$$

By Remark 3.2, it implies that

$$\rho(Gx_{n+1}, Gu^*) \leq \frac{\rho(Gx_n, Gx_{n+1}) + \rho(Gx_{n+1}, Gu^*)}{2}. \tag{3.50}$$

Applying (3.50) in (3.49) we obtain

$$\rho(u^*, Gu^*) \leq \rho(u^*, Gx_{n+1}) + \frac{\rho(Tx_n, Gx_{n+1}) + \rho(Gx_{n+1}, Gu^*)}{2}.$$

Letting $n \to \infty$ and using in above inequality, we get

$$\begin{aligned}
\rho(u^*, Gu^*) &\leq \rho(u^*, Gu^*) + \frac{\rho(u^*, Gu^*) + \rho(Gu^*, Gu^*)}{2}, \\
\rho(u^*, Gu^*) &\leq \rho(u^*, Gu^*) + \frac{\rho(u^*, Gu^*)}{2}, \\
\rho(u^*, Gu^*) &\leq \frac{2\rho(u^*, Gu^*) + \rho(u^*, Gu^*)}{2}, \\
2\rho(u^*, Gu^*) &\leq 2\rho(u^*, Gu^*) + \rho(u^*, Gu^*), \\
2\rho(u^*, Gu^*) - 2\rho(u^*, Gu^*) &\leq d(u^*, Tu^*), \\
0 &\leq \rho(u^*, Gu^*),
\end{aligned}$$

which is a contradiction. Hence $Gu^* = u^*$.

Next, using (iv) we prove that u^* is a unique fixed point of G. Assume on contrary, that there exists $v^* \in int(P)$ such that $u^* \neq v^*$ or $Gu^* \neq Gv^*$. Let $Gx_n \to v^*$ and v^* is a fixed point of G. Using Remark 3.2 and (3.35), it follows that $u^* = v^*$ or $Gu^* = Gv^*$ which is a contradiction. Thus, u^* is a unique fixed point of G.

Moreover, G is a subsequentially convergent, using (v), $\{f^n x_0\}$ has a convergent subsequence, there exists $v^* \in X$ and $\{f^{n_k} x_0\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} f^{n_k} x_0 = v^*.$$

Due to the continuity of G, using (d) it implies that

$$\lim_{k\to\infty} G f^{n_k} x_0 = Gv^*.$$

By (3.48), we conclude that

$$Gv^* = u^*.$$

Using Remark 3.2, we get

$$\begin{aligned}
\rho(Gff^{n_{k-1}}x_0, Gff^{n_k}x_0) &\le \lambda(\rho(Gf^{n_k-1}x_0, Gff^{n_k-1}x_0) + \rho(Gf^{n_k}x_0, Gff^{n_k}x_0),\\
&\le \lambda(\rho(Gf^{n_k-1}x_0, Gff^{n_k-1}x_0) + \rho(Gff^{n_{k-1}}x_0, Gff^{n_k}x_0),\\
&\le \lambda\rho(Gf^{n_k-1}x_0, Gff^{n_k-1}x_0) + \lambda\rho(Gff^{n_{k-1}}x_0, Gff^{n_k}x_0),\\
&\le \frac{\lambda}{1-\lambda}\rho(Gf^{n_k-1}x_0, Gff^{n_{k-1}}x_0). \qquad (3.51)
\end{aligned}$$

Thus, using (*iii*) of Definition 2.18, we have

$$\rho(Gfv^*, Gv^*) \le \rho(Gfv^*, Gf^{n_k+1}x_0) + \rho(Gf^{n_k+1}x_0, Gv^*), \qquad (3.52)$$

By Remark 3.2,

$$\begin{aligned}
d\rho(Gfv^*, Gf^{n_k+1}x_0) &= d(Gfv^*, Gff^{n_k}x_0)\\
&\le \lambda\Big[\rho(Gv^*, Gfv^*) + \rho(Gff^{n_{k-1}}x_0, Gff^{n_k}x_0)\Big]. \qquad (3.53)
\end{aligned}$$

Using (3.53) and (3.51) in (3.52), we obtain

$$\begin{aligned}
\rho(Gfv^*, Gv^*) &\le \lambda\Big[\rho(Gv^*, Gfv^*) + \rho(Gff^{n_{k-1}}x_0, Gff^{n_k}x_0)\Big] + \rho(Gf^{n_k+1}x_0, Gv^*),\\
&\le \lambda\Big[d\rho(Gv^*, Gfv^*) + \frac{\lambda}{1-\lambda}\rho(Gf^{n_k-1}x_0, Gff^{n_{k-1}}x_0)\Big]\\
&\quad + \rho(Gf^{n_k+1}x_0, Gv^*),\\
&\le \lambda\rho(Gv^*, Gfv^*) + \lambda\Big(\frac{\lambda}{1-\lambda}\Big)^{n_k-1}\rho(Gf^{n_k-1}x_0, Gff^{n_{k-1}}x_0)\\
&\quad + \rho(Gf^{n_k+1}x_0, Gv^*),\\
&\le \frac{\lambda}{1-\lambda}\Big(\frac{\lambda}{1-\lambda}\Big)^{n_k-1}\rho(Gf^{n_k-1}x_0, Gff^{n_k-1}x_0) +\\
&\quad \frac{1}{1-\lambda}\rho(Gf^{n_k+1}x_0, Gv^*),\\
&\le \Big(\frac{\lambda}{1-\lambda}\Big)^{n_k}\rho(Gf^{n_k-1}x_0, Gff^{n_k-1}x_0)\\
&\quad + \frac{1}{1-\lambda}\rho(Gf^{n_k+1}x_0, Gf^{n_k}x_0). \qquad (3.54)
\end{aligned}$$

Suppose that

$$\begin{aligned}
\Big(\frac{\lambda}{1-\lambda}\Big)^{n_k}\rho(Gf^{n_k-1}x_0, Gff^{n_k-1}x_0) &= \frac{c}{2}. \qquad (3.55)\\
\frac{1}{1-\lambda}\rho(Gf^{n_k+1}x_0, Gf^{n_k}x_0) &= \frac{c}{2}. \qquad (3.56)
\end{aligned}$$

Letting $k \to \infty$ and using Definition 3.3, (3.55) and (3.56) in (3.54), we obtain

$$\rho(Gfv^*, Gv^*) \le \frac{c}{2} + \frac{c}{2},$$

which follows

$$\rho(Gfv^*, Gv^*) \;\leq\; c.$$

Since G is one to one and continuous, $fv^* = v^*$. So, f has a fixed point. As $Gf^n x_0$ is sequentially convergent, we conclude that $\{Gf^n x_0\}$ converges to the fixed point of f. □

Next, we prove our second main result by extending Theorem 3.3 using an F-Kannan-Suzuki type mapping in TVS-valued cone metric space.

Theorem 3.6 *Let (X,ρ) be a complete TVS-valued cone metric space and P a solid cone. In addition let $G: X \to X$ be a one-to-one, continuous and sequentially convergent mapping and $f: X \to X$ be such that $F(f) \neq \emptyset$, $\vartheta > 0$ and that*

$$\begin{aligned} \frac{1}{2}\rho(x,Gx) &< \rho(x,y) \\ \implies \vartheta + F(\rho(Gfx, Gf^2x)) &\leq F(\rho(Gx,Gfx)), \end{aligned} \tag{3.57}$$

holds for some $\lambda \in (0,1)$ and for all $x \in X, x \neq fx$. Then f has property Q.

Proof. By Definition 3.6, let $u \in F(G^n) \cap F(f^n)$ for some $n \in \mathbb{N}$. If $u = fu$, then u is a unique fixed point of G and f. Hence, the proof is completed. On contrary, we suppose $u \neq fu$. Let $x = u = f^{n-1}u$ and $y = fu = ff^{n-1}u$ such that $f^{n-1} \neq ff^{n-1}$ and using (3.57), we get

$$\begin{aligned} \frac{1}{2}\rho(u,Gu) &< \rho(u,fu), \\ \rho(u,Gu) &< 2\rho(u,fu), \\ \implies \vartheta + F\Big[\rho(Gff^{n-1}u, Gf^2f^{n-1}u)\Big] &\leq F\Big[\rho(\rho f^{n-1}u, Gff^{n-1}u)\Big], \\ \vartheta + F\Big[\rho(Gff^{n-1}u, Gff^nu)\Big] &\leq F\Big[\rho(Gf^{n-1}u, Gf^nu)\Big], \end{aligned}$$

Consequently, we get

$$F\Big[\rho(Gff^{n-1}u, Gff^nu)\Big] \;\leq\; F\Big[d(Gf^{n-1}u, Gf^nu)\Big] - \vartheta,$$

Repeating the same argument several times, we finally obtain

$$F\Big[\rho(Gff^{n-1}u, Gff^nu)\Big] \;\leq\; F\Big[\rho(Gf^{n-1}u, Gf^nu)\Big] - n\vartheta.$$

By following similar procedure as the proof of Theorem 3.5, we can conclude that $\rho(Gu, Gfu) = c$, i.e., $Gu = Gfu$. Since G is one to one and sequentially convergent, then $u = fu$, which is a contradiction. Hence $u \in F(G^n) \cap F(f^n)$. □

We give an example where generalised Kannan mapping will not be applicable. However, F-Kannan-Suzuki type mapping is applicable.

Example 3.18 Consider the sequence $X = \{0,1\} \cup \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ and d be an Euclidean metric on X. Then (X,d) is a TVS-valued cone complete metric space. Let the mapping $f : X \to X$ be determined as follows:

$$\begin{aligned} f(0) &= 0 \\ f(1/n) &= \frac{1}{n+1}. \end{aligned}$$

for $n \geq 2$. Let there exists $\lambda \in [0, \frac{1}{2})$, so that for all $x,y \in X$ condition (2.1) is satisfied although is not true for every $\lambda > 0$. That is a contradiction, hence Kannan's theorem can not be applicable.

Next, let the mapping $G : X \to X$ be determined as

$$\begin{aligned} G(0) &= 0 \\ G(1/n) &= \frac{1}{2^n}. \end{aligned}$$

Here, for all $n \geq 2$, G is continuous, one to one, and subsequentially convergent.

We consider a sequence $\{x_n\}$ in X and assume that X is sequentially compact in complete TVS-valued cone metric space. By assumption X is sequentially compact with $\varepsilon = 1$ we can cover the space X with finitely many balls of radius 1; then one of them contain's many $\{x_n\}$ for $n \geq 2$; i.e,. There is a ball B_1 of radius 1 so that there is a subsequence of $\{x_n\}$ whose members all belongs to B_1. We denote this subsequence by $\{x_{n_1}\}$ thus all $\{x_{n_1}\}$ belongs to B_1.

Similarly by sequentially compactness conditions with $\varepsilon = \frac{1}{2}$, we can find a subsequence $\{x_{n_2}\}$ of $\{x_{n_1}\}$ and a ball B_2 of radius $\frac{1}{2}$ so that all $\{x_{n_2}\}$ belongs to B_2. Continuing this way, we obtain for any $k \geq 2$ a subsequence $\{x_{n_k}\}$ of $\{x_{n_k - 1}\}$ and a ball B_k of radius 2^{-k} so that all $\{x_{n_k}\}$ belongs to B_k.

Now, let $m,n \in \mathbb{N}$, $m > n$. Then, we prove that (f,G) is a F-Kannan-Suzuki type mapping in TVS -valued cone metric space with respect to $F_2(z) = -\frac{1}{z}$ and $\vartheta \geq 0$. By using $(FKS2)$ and $F_2(z)$ we have

$$\begin{aligned} \frac{1}{2}\rho(x,Gx) &< \rho(x,y) \\ \Longrightarrow \frac{\rho(Gx,Tfx)+\rho(Gy,Gfy)}{2} &\leq \frac{\rho(Gfx,Gfy)}{1-\vartheta\rho(Gfx,Gfy)}. \end{aligned} \tag{3.58}$$

To see this, we now calculate $\rho(fx,Gy)$ for $x=\frac{1}{m}, y=\frac{1}{n}, n\geq 1$.

$$\begin{aligned}
\rho(Gx,Gfx) &= \rho(G(1/n),Gf(1/n)),\\
&\leq \left|\frac{1}{2^n}-\frac{1}{2^{\frac{1}{n+1}}}\right|e^t. && (3.59)\\
\rho(Gfx,Gfy) &= \rho(Gf(1/n),Gf(1/m)),\\
&\leq \left|2^{\frac{1}{n+1}}-2^{\frac{1}{m+1}}\right|e^t. && (3.60)\\
\rho(Gy,Gfy) &= \rho(G(1/m),Gf(1/m)),\\
&\leq \left|2^{\frac{1}{m}}-2^{\frac{1}{m+1}}\right|e^t. && (3.61)
\end{aligned}$$

Applying $(3.59),(3.60)$ and (3.61) in (3.58) becomes

$$\begin{aligned}
\frac{1}{2}\rho(x,Gx) &< \rho(x,y)\\
\frac{1}{2}\rho(1/n,G(1/n)) &< \rho(1/n,1/m)\\
\rho(1/n,G(1/n)) &< 2\rho(1/n,1/m)\\
\left|\frac{1}{n}-\frac{1}{2^n}\right|e^t &< 2\left|\frac{1}{n}-\frac{1}{m}\right|e^t\\
\left|\frac{2^n-n}{2^n.n}\right|e^t &< 2\left|\frac{m-n}{n}m\right|e^t\\
\Longrightarrow \frac{\rho(G(1/n),Gf(1/n))+\rho(G(1/m),Gf(1/m))}{2} &\leq \frac{\rho(Gf(1/n),Gf(1/m))}{1-\vartheta\rho(Gf(1/n),Gf(1/m))}
\end{aligned}$$

$$\frac{\left|\frac{1}{2^n}-\frac{1}{2^{\frac{1}{n+1}}}\right|e^t+\left|2^{\frac{1}{m}}-2^{\frac{1}{m+1}}\right|e^t}{2} \leq \frac{\left|2^{\frac{1}{n+1}}-2^{\frac{1}{m+1}}\right|e^t}{1-\vartheta\left|2^{\frac{1}{n+1}}-2^{\frac{1}{m+1}}\right|e^t}.$$

The inequality (3.58) and all conditions imposed in Theorem 3.5 are satisfied. Hence, G and f has unique fixed point that is $v^*=0$ in $\{P\subseteq E\}\in X$, where P is a solid cone.

3.5 Some Applications

In this section, we will provide five applications of the theorems proved in the previous sections.

3.5.1 Existence of a Solution for an Integral Equation

In this section as motivated by Nashine *et al.* (2011), we establish an application of Theorem 3.4 to get a common solution of the following Volterra type integral equation:

This problem is equivalent to the integral equation:

$$u(t) \;=\; h(t)+\int_a^b f(t,s,u(s))ds, \forall\, t,s\in[a,b]. \tag{3.62}$$

where $f:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ and $h:[a,b]\to\mathbb{R}$ are continuous functions. Let $X=C([a,b],\mathbb{R})$ be the space of all continuous functions defined on $C[a,b]$. Notice that $C[a,b]$ endowed with metric

$$d(x,y)=||x-y||_\infty=\max_{t\in[a,b]}|x(t)-y(t)|, \tag{3.63}$$

is a complete metric space and X can be equipped with the partial order $\preceq$ given by $x,y\in X, (x\preccurlyeq y) \implies (x(t)\preceq y(t)$ and $||x||_\infty,||y||_\infty\leq 1)$, or $(x(t)=y(t))$ for all $t\in[a,b]$. It was shown by Nieto and Rodrigurz-Lopez (2005) that $(X,\preceq)$ is regular. For more applications to non linear integral equations one can see (Hussain and Adeel 2019, Kanwal *et al.* 2019, Zhitao 1996) and the references therein.

Now, we define a mapping $S:X\to X$ by

$$Sx(t)=h(t)+\int_a^t f(t,s,x(s))ds,\; t\in[a,b]. \tag{3.64}$$

If $x\in[a,b]$ is a fixed point of S, then $x\in[a,b]$ is a solution of (3.62).

We prove our results by establishing the existence of a common fixed point for a pair of self mappings:

Theorem 3.7 *Let $S,T:C([a,b])\longrightarrow C([a,b])$ be self maps of a metric space (X,d) such that the following conditions hold:*

(1) *for all $t,s\in[a,b]$ and $x\in C[a,b]$, we have*

$$f(t,s,u(t)) \;\leq\; f\left(t,s\int_a^b f(s,\tau,u(\tau))d\tau+h(t)\right),$$

(2) *there exist two functions $f_1,f_2:[a,b]\times X\times X\longrightarrow X$ with constants α such that, for all $t\in[a,b]$, we have*

$$|f_1(t,s,x)|-|f_2(t,s,y)| \;\leq\; e^{-\Upsilon}\alpha(t,s)q|x-y|,$$

$q\leq 1$,

(3) *for* $t,s \in [a,b]$ *we have*

$$\sup_{t\in[a,b]} \int_a^b \alpha(t,s)ds \leq 1.$$

(4) *there exists* $T \in S$ *and* $\mathscr{M}$ *such that*

$$\mathscr{M}(x,y) = \frac{d(Tx,TSx)+d(Ty+TSy)}{2},$$

$\forall x,y \in X.$

Then the integral equation (3.62) *has a solution* $x^* \in C([a,b],\mathbb{R})$.

Proof. From (1), for all $t,s \in [a,b]$, we have

$$\begin{aligned} Sx(t) &= h(t)+\int_a^b f(t,s,x(s))ds, \forall\, t,s \in [a,b]. \\ &\leq \int_a^b f\left(t,s\int_a^b f(s,\tau,x(\tau))d\tau + h(s)\right)ds + h(t) \\ &= h(t)+\int_a^b f(t,s,Sx(s))ds, \forall\, t,s \in [a,b] \\ &= S(Sx)(t). \end{aligned} \tag{3.65}$$

Therefore, $Sx \leq S(Sx)$ for all $x \in [a,b]$.

By using conditions (2), (3) and (4) of Theorem 3.7, we obtain

$$\begin{aligned} |Sx(t)-Sy(t)| &= \int_a^b |f_2(t,s,x(s))-f_1(t,s,y(s))|ds, \\ &\leq e^{-\Upsilon}\int_a^b |\alpha(t,s)q|ds(\alpha|x(t)-y(t)|), \\ &= e^{-\Upsilon}q\mathscr{M}(x,y)\int_a^b |\alpha(t,s)|ds, \\ &\leq e^{-\Upsilon}q\mathscr{M}(x,y). \end{aligned}$$

By taking the natural logarithms on both sides of the above inequality and the property of F with $q \leq 1$, we get

$$\begin{aligned} \Upsilon + F(d(Sx,Sy)) &\leq F(\mathscr{M}(x,y)), \\ \textit{where} & \\ \mathscr{M}(x,y) &= \frac{d(Tx,TSx)+d(Ty+TSy)}{2}, \end{aligned}$$

for all $x,y \in X$. Hence $x = x^*$ is a common fixed point of S and T, also a solution to integral equation. Then, the integral equation (3.62) has a solution $x^* \in C([a,b],\mathbb{R})$. □

3.5.2 Existence of a Solution for Nonlinear Fractional Differential Equation

The purpose of this section is to provide an application of Theorem 3.4 to get a common solution of a nonlinear fractional differential equation, where we can apply F-Kannan mappings in metric spaces.

Here, we investigate the Caputo derivative with the fractional order of the non-linear fractional differential equation. This form of fractional derivative for a continuous function $f : [0,\infty) \to \mathbb{R}$ is given as:

$$({}^{\mathrm{C}}\mathrm{D}_{\mathrm{t}}^{\alpha})\mathrm{f}(\mathrm{t}) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_{\mathrm{a}}^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\mathrm{n}-\alpha-1}\mathrm{f}^{\mathrm{n}}(\mathrm{s})\mathrm{ds}, (\mathrm{n}-1 < \alpha, \mathrm{n} = [\alpha]+1),$$

where $[\alpha]$ denotes the integer part of the real number α (see (Baleanu *et al.* 2013, Kanwal *et al.* 2019)). Also, the Riemann-Liouville fractional integral of order α is given by

$$(I_s^{\alpha})f(t) = \frac{1}{\Gamma(\alpha)-1} \int_0^t (t-s)^{\alpha} f(s)ds \; (\alpha > 0).$$

The Caputo fractional differential equation has several applications in mathematics, i.e., in image processing, Digital data processing, electrical signal, acoustics, physics and probability theory (one can see in Zölzer *et al.* (2002)). The following nonlinear fractional differential equation is inspired by Delbosco and Rodino (1996), Kilbas *et al.* (2006) and Budhia *et al.* (2020).

$$\begin{cases} {}^{C}D_s^{\alpha} x(t) = f(t, x(t)), \; t \in (0,1), 1 < \alpha \leq 2, \\ \\ x(0) = 0, x(1) = \int_0^{\nu} x(s)ds \; (0 < \nu < 1) \end{cases} \tag{3.66}$$

where ${}^{C}D_s^{\alpha}$ denotes the Caputo fractional derivative of order α and $f : [0,1] \to X$ is a continuous function.

Consider the space $X = C(I)(I = [0,1])$ of the continuous function defined on I. Suppose that $(X, \|.\|)$ is a Banach space, and $I := [0,T], T > 0$. Let $C(I,X)$ be the Banach space of all continuous functions from I into X with the norm $\|x\| := \sup |x(t)| = L, t \in I$ for $x \in C(I,X)$ (one can see in (Zhou 2016).

This space defines the metric by

$$d(x,y) = \sup_{t\in[0,T]} \{|x(t) - y(t)|\} \tag{3.67}$$

$\forall x, y \in X$. This is a complete metric space.

The nonlinear fractional Equation 3.66 can be written as follows.

$$\begin{aligned} \text{x(t)} \;=\; & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds - \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,x(s))ds + \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)}\int_0^\nu \left[\int_0^s (s-z)^{\alpha-1} f(z,x(z))dz\right] ds. \end{aligned} \tag{3.68}$$

A function $x \in C(I,X)$ is a solution of the fractional differential integral equation (3.66) if and only if x is a solution of the non-linear fractional differential equation (3.68).

Now, we prove the following theorem.

Theorem 3.8 *Suppose the following conditions hold:*

(i) $f \in C(I\times X, X)$ *is sequentially continuous;*

(ii) *there exists a continuous function* $f : [0,1]\times\mathbb{R} \to \mathbb{R}_+$, *such that*

$$|f(t,x(s) - f(t,y(s)| \leq e^{-\Upsilon} L|x(s) - y(s)|,$$

for all $t \in [0,1]$ *and for all* $x, y \in X$ *such that* $d(x(t), y(t)) \geq 0$ *and a constant* L *and* κ *such that*

$$L\kappa \leq 1,$$

and

$$\kappa = \frac{t^\alpha(2-\nu^2)(\alpha+1) + 2t(\alpha+\nu^{\alpha+1}+1)}{(2-\nu^2)\Gamma(\alpha)\alpha(\alpha+1)}.$$

Then, the fractional differential equation (3.66) *has a common solution as a fixed point* $x^* \in C(I,X)$.

Proof. Let us define $T,S : C(I) \to C(I)$ by

$$\begin{aligned} \text{TSx(t)} \;=\; & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds - \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,x(s))ds + \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)}\int_0^\nu \left[\int_0^s (s-z)^{\alpha-1} f(z,x(z))dz\right] ds, \end{aligned} \tag{3.69}$$

for $t \in [0,1]$, then TS is sequentially continuous. Suppose that

$$Sx(t) = \int_0^s (s-z)^{\alpha-1} f(z,x(z))dz,$$

this implies that $S \in TS$ and S posses a fixed point $x^* \in TS$. To prove the existence of fixed point of TS, we prove that TS is sequentially continuous and contraction. To show TS is sequentially continuous, let $TSx \neq TSy$, for all $x,y \in [0,T]$. By condition (ii), we have

$$\begin{aligned} |TSx - TSy| \;&=\; \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds - \\ &\quad \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,x(s))ds + \\ &\quad \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^\nu \left[\int_0^s (s-z)^{\alpha-1} f(z,x(z))dz \right] ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds \end{aligned}$$

$$\begin{aligned} &+ \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,y(s))ds - \\ &\frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^\nu \left[\int_0^s (s-z)^{\alpha-1} f(z,y(z))dz \right] ds \Bigg|, \\ \leq\; & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| ds + \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |f(s,x(s)) - f(s,y(s))| ds, + \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^\nu \left[\int_0^s (s-z)^{\alpha-1} |f(z,x(z)) - f(z,y(z))| dz \right] ds, \\ \leq\; & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)| ds + \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |x(s) - y(s)| ds + \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^\nu \left[\int_0^s (s-z)^{\alpha-1} |x(z) - y(z)| dz \right] ds, \end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-\Upsilon}L}{\Gamma(\alpha)}\|x-y\|_\infty\int_0^t (t-s)^{\alpha-1}ds+ \\
&\quad \frac{2te^{-\Upsilon}L}{(2-\nu^2)\Gamma(\alpha)}\|x-y\|_\infty\int_0^1 (1-s)^{\alpha-1}ds+ \\
&\quad \frac{2te^{-\Upsilon}L}{(2-\nu^2)\Gamma(\alpha)}\|x-y\|_\infty\int_0^\nu \left[\int_0^s (s-z)^{\alpha-1}dz\right]ds, \\
&\leq \left[\frac{e^{-\Upsilon}Lt^\alpha}{\alpha\Gamma(\alpha)}+\frac{2te^{-\Upsilon}L}{(2-\nu^2)\alpha\Gamma(\alpha)}+\frac{2te^{-\Upsilon}L\nu^{\alpha+1}}{(2-\nu^2)\alpha(\alpha+1)\Gamma(\alpha)}\right]\|x-y\|_\infty,
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{-\Upsilon}L\left[\frac{t^\alpha}{\alpha\Gamma(\alpha)}+\frac{2t}{(2-\nu^2)\alpha\Gamma(\alpha)}+\frac{2t\nu^{\alpha+1}}{(2-\nu^2)\alpha(\alpha+1)\Gamma(\alpha)}\right]\|x-y\|_\infty, \\
&= e^{-\Upsilon}L\left[\frac{t^\alpha(2-\nu^2)(\alpha+1)+2t(\alpha+\nu^{\alpha+1}+1)}{(2-\nu^2)\Gamma(\alpha)\alpha(\alpha+1)}\right]\|x-y\|_\infty, \\
&\leq e^{-\Upsilon}L\kappa\|x-y\|_\infty. \qquad (3.70)
\end{aligned}
$$

This implies that

$$\|TSx,TSy\|_\infty \leq e^{-\Upsilon}L\kappa\|x-y\|_\infty.$$

Since $L\kappa < 1$, we have

$$\|TSx,TSy\|_\infty \leq e^{-\Upsilon}\|x-y\|_\infty.$$

Thus, for each $x,y \in X$, we have

$$d(TSx,TSy) \leq e^{-\Upsilon}\mathbb{M}(x,y). \qquad (3.71)$$

Taking logarithms on both sides of (3.71) using $F_1(z) = \ln(z)$ and the property of F, we get

$$\ln(d(TSx,TSy)) \leq \ln(e^{-\Upsilon}\mathbb{M}(x,y)).$$

Equivalently to

$$\Upsilon + F(d(TSx,TSy)) \leq F(\mathbb{M}(x,y)).$$

For $L\kappa \in [0,1)$, $\Upsilon > 0$ and $\mathbb{M}(x,y)$ is an F-Kannan mapping. Therefore, TS is a contraction mapping on X. Since all the conditions of Theorem 3.8 are satisfied. Therefore, there exists $x^* \in C(I)$ a common fixed point of T and S, that is, x^* is a solution to fractional nonlinear differential equation (3.66). □

3.5.3 Existence of Common Solution of Ordinary Differential Equations for Damped Forced Oscillations

This section investigates the solution of the forced damped oscillations differential equations problem in the setting of metric spaces. Nieto and Rodriguez (2005) initiated the proof of the existence of a solution of an ordinary differential equation. Since then, several authors are interested in this line of research. In details, one can see the literature in Borisut *et al.* (2019), Gupta *et al.* (2017), Harjani and Sadarangani (2010), Kilbas *et al.* (2006), X *et al.* (2019), Yan *et al.* (2012) and the references therein.

Shoaib *et al.* (2020), considered the forced damped oscillations differential problem of an object of mass m moving to and fro on the x-axis around an equilibrium position $x = 0$. The object has position $x(t)$ at time t, it undergo the applied force $f(t)$ such that

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t). \tag{3.72}$$

where m, b, k are constant positive numbers. If the intial conditions are assumed to be

$$x(0) = 0,\ x'(0) = 0,$$

then problem (3.72) can be written as

$$x(t) = \int_0^T G(t,s)K(t,x(s))ds, \forall\, t,s \in [0,T], \tag{3.73}$$

where $T > 0$. Let $X = C(I)(I = [0,T])$.

The green function for forced damped oscillation is defined by

$$G(t,s) = \begin{cases} -se^{\tau(T+s-t)}, & 0 \le s \le t \le T, \\ -te^{\tau(t-s)}, & 0 \le t \le s \le T, \end{cases}$$

where τ can be written in terms of b, k and m.

Inspired by Shoaib *et al.* (2020) and Gupta *et al.* (2017), we find the common solution of a forced damped oscillations differential equation using fixed point method.
Theorem 3.9 *Suppose the following assumptions hold:*

(i) *there exists a continuous function* $K : [0,T]^2 \to [0,\infty)$, *such that*

$$|K(t,s,x(t)) - K(t,s,y(t))| \leq e^{-\Upsilon}\sqrt{\frac{\ln[(x-y)^2+1]}{x-y}},$$

for $t,s \in [0,T]$, $\Upsilon > 0$ *and* $x,y \in \mathbb{R}$, *where;*

$$\sqrt{\frac{\ln[(x-y)^2+1]}{x-y}} = \frac{d(Tx,TSx)+d(Ty,TSy)}{2},$$

(ii) *there exists a functon* $G : I \times I \to \mathbb{R}_+$, *such that*

$$\sup_{t\in[0,T]} \int_0^T G^2(t,s)ds \leq \frac{1}{T}.$$

Then, the problem (3.73) *has a fixed point* $x^* \in X$, *which is a solution of* (3.72).

Proof: Let $TS : C^1([0,1]) \to C^1([0,1])$ be an operator defined by

$$TSx(t) = \int_0^T G(t,s)K(t,s,x(s))ds. \tag{3.74}$$

Consider $x > y$, for all $x,y \in C(I)$ and (3.73), we have

$$\begin{aligned} |TSx(t) - TSy(t)| &= \sup_{t\in[0,1]} \int_0^T G(t,s)[K(t,s,x(s)) - K(t,s,y(s))]ds, \\ &\leq \sup_{t\in[0,T]} \int_0^t e^{-\Upsilon}G(t,s)\sqrt{\frac{\ln[(x(s)-y(s))^2+1]}{x(s)-y(s)}}ds. \end{aligned} \tag{3.75}$$

Recall Cauchy-Schwartz inequality defined by

$$\left(\sum_{i=1}^{\infty} |x_i y_i|\right)^2 \leq \left(\sum_{i=1}^{\infty} |x_i|^2\right)\left(\sum_{i=1}^{\infty} |y_i|^2\right). \tag{3.76}$$

Applying (3.76) in (3.75), we obtain

$$\left(\int_0^T e^{-\Upsilon}G(t,s)\sqrt{\frac{\ln[(x(s)-y(s))^2+1]}{x(s)-y(s)}}\right) \leq$$

$$\left(\int_0^T G^2(t,s)\right)^{\frac{1}{2}}\left(\int_0^T \left[e^{-2\Upsilon}\sqrt{\frac{\ln[(x(s)-y(s))^2+1]}{x(s)-y(s)}}\right]^2\right)^{\frac{1}{2}}. \tag{3.77}$$

Then

$$\int_0^T G^2(t,s) \;=\; \frac{1}{T}. \tag{3.78}$$

The second integral gives

$$\int_0^T \left[e^{-2\Upsilon}\frac{\ln[(x(s)-y(s))^2+1]}{x(s)-y(s)}\right]ds \;\leq\; e^{-2\Upsilon}\frac{\ln[(\|x(s)-y(s)\|)^2+1]}{\|x(s)-y(s)\|}\times T. \tag{3.79}$$

Using (3.78) and (3.79) in the R.H.S of (3.77), we get

$$\leq \;\left(\frac{1}{T}\right)^{\frac{1}{2}}\left(e^{-2\Upsilon}\frac{\ln[(d(x,y))^2+1]}{d(x,y)}\times T\right)^{\frac{1}{2}}.$$

Hence from (3.75),

$$d(TSx,TSy) \;\leq\; e^{-\Upsilon}\sqrt{\frac{\ln[(d(x,y))^2+1]}{d(x,y)}}. \tag{3.80}$$

Taking logarithms on both sides of (3.80) using $F_1(z) = \ln(z)$ and the property of F, we get

$$\ln(d(TSx,TSy) \;\leq\; \ln(e^{-\Upsilon}\sqrt{\frac{\ln[(d(x,y))^2+1]}{d(x,y)}}). \tag{3.81}$$

This implies that

$$\Upsilon + F(d(TSx,TSy)) \;\leq\; F\left(\sqrt{\frac{\ln[(d(x,y))^2+1]}{d(x,y)}}\right).$$

Let us choose a map $Sx = \ln[(d(x,y))^2+1]$, it is obvious, $S \in TS$.

Thus, we conclude that for $x > y$ all conditions of Theorem 3.9 are satisfied. Hence TS has a unique common fixed point x^* which is the solution of integral equation (3.73).

3.5.4 Existence of a Solution for Nonlinear Riemann-Liouville Type Fractional Differential Equation

The nonlinear fractional differential equation is used as convolution mapping. Convolution and associated functions are found in many applications in science, engineering, and mathematics.

(i) **In image processing**: In digital image processing, convolutional filtering plays an important role in many important algorithms in edge detection and related processes. In optics, an out-of-focus photograph is a convolution of the sharp image with a lens function. The photographic term for this is bokeh. In image processing applications, such as adding blurring.

(ii) **In digital data processing**: In analytical chemistry, Savitzky-Golay smoothing filters are used for the analysis of spectroscopic data. They can improve the signal-to-noise ratio with minimal distortion of the spectra. In statistics, a weighted moving average is a convolution.

(iii) **In acoustics**: Reverberation is the convolution of the original sound with echoes from objects surrounding the sound source. In digital signal processing, convolution is used to map the impulse response of a real room on a digital audio signal. In electronic music, convolution is the imposition of a spectral or rhythmic structure on a sound. Often this envelope or structure is taken from another sound. The convolution of two signals is the filtering of one through the other.

(iv) **In electrical engineering**: The convolution of one function (the input signal) with a second function (the impulse response) gives the output of a linear time-invariant system (LTI). At any given moment, the output is an accumulated effect of all the prior values of the input function, with the most recent values typically having the most influence (expressed as a multiplicative factor). The impulse response function provides that factor as a function of the elapsed time since each input value occurred.

(v) **In physics**: Wherever is a linear system with a "superposition principle," a convolution operation makes an appearance. For instance, in spectroscopy, line broadening due to the Doppler effect on its own gives a Gaussian spectral line shape and collision broadening alone gives a Lorentzian line shape. When both effects are operative, the Line shape is a convolution of Gaussian and Lorentzian, a Voigt function. In time-resolved fluorescence spectroscopy, the excitation signal can be treated as a chain of delta pulses and the measured fluorescence is a sum of exponential decays from each delta pulse

(v) **In computational fluid dynamics**: The large eddy simulation (LES) turbulence the model uses the convolution operation to lower the range of length scales necessary in computation, thereby reducing the computational cost.

(vi) **In probability theory**: The probability distribution of the sum of two independent random variables is the convolution of their distributions.

(vi) **In kernel density estimation**: A distribution is estimated from sample points by convolution with a kernel, such as an isotropic Gaussian.

(vii) **In radiotherapy**: Most parts of all modern codes of calculation apply the convolution-superposition algorithm in the treatment of planning systems.

The above applications of a convolution show that the fractional derivative as convolution has multiple purposes. It can portray the memory, as in the case of the theory of elasticity. Second, it can be considered as a filter. In particular, the Caputo and Caputo-Fabrizio type can be viewed as a a filter of the local derivative with power and exponent functions (one can see in Zölzer *et al.* (2002)).

The purpose of this section is to provide an application of Theorem 3.5 to get a common solution of a nonlinear fractional differential equation, where we can apply F-Kannan-Suzuki type mappings in complete TVS valued complete cone metric spaces.

Here, we investigate the Riemann-Liouville derivative fractional integral of order $\alpha > 0$. This form of fractional derivative for a continuous function $g : [0,\infty) \to \mathbb{R}$ denoted by $D_a^{\alpha} f$, is given by

$$(\mathrm{D}^{\alpha}_{0+})\mathrm{g}(\mathrm{t}) = \left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^{\mathrm{n}-1}(\mathrm{I}^{\alpha}_{0+})\mathrm{g}(\mathrm{t}) \quad = \quad \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1} g(s)ds,$$

where $[\alpha]$ denotes the integer part of the real number α and $n = [\alpha]+1$, provided that the right hand side is pointwise defined on $(0,\infty)$. (see Podlubny (1999), Baleanu *et al.*

(2013), Cabada and Wang (2012), Henderson and Luca (2016), Zhou (2016), Kanwal *et al.* (2019)). Also, the Riemann-Liouville fractional integral of order α is given by

$$(I_0^\alpha)g(t) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)ds,$$

for $\alpha > 0$. The notation $[\alpha]$ stands for largest interger not greater than α. If $\alpha = m \in \mathbb{N}$, then $(D_{0+}^m)g(t) = g^m(t)$, for $t > 0$ and if $\alpha = 0$, then $(D_{0+}^0)g(t) = g(t)$ for $t > 0$.

The following nonlinear fractional differential equation with integral boundary valued conditions is inspired by Kilbas *et al.* (2006), Cabada and Hamdi (2014), Cabada and Wang (2012).

$$\begin{cases} D_{0+}^\alpha x(t) + g(t, x(t)) = 0, \ 0 < t < 1, \\ \\ x(0) = x'(0), \\ \\ x'(1) = \lambda \int_0^1 x(s)ds, \ 0 < \lambda < 1, \end{cases} \tag{3.82}$$

where D_{0+}^α denotes the Riemann-Liouville fractional derivative of order α and $g : [0,1] \to X$ is a continuous function.

We recall the following lemmas from Bai and Lu (2005).

Lemma 3.2 *Let $\alpha > 0$. If we assume $x \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^\alpha x(t) = 0,$$

has

$$x(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

$C_i \in \mathbb{R}$, $i = 1, 2, \ldots N$, *as unique solution.*

For $p \geq 1$ and $d \in \mathbb{N}$, $L^p = L^p((a,b), \mathbb{R}^d)$ denote the classical Lebesgue space of p-integrable functions endowed with its usual norm $\|.\|_{L^p}$. We denote $|.|$ the Euclidean norm of $\mathbb{R}^d$ and $C = C((a,b),\mathbb{R}^d)$ the space of continuous functions endowed with the usual norm $\|.\|$.

Since $D^{\alpha}_{0+}I^{\alpha}_{0+}x(t) = x$ for all $x \in C(0,1)\cap L(0,1)$. From Lemma 3.2 we deduce the following lemma.

Lemma 3.3 *Assume that* $x \in C(0,1)\cap L(0,1)$, *with fractional derivative of order* $\alpha > 0$ *that belongs to* $x \in C(0,1)\cap L(0,1)$. *Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}x(t) = x(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

for some $C_i \in \mathbb{R}$, $i = 1,2,\dots N$, *as unique solution.*

The unique solution of (3.82) is given by

$$x(t) \;=\; \int_a^t G_f(t,s)g(s,u(s))ds. \tag{3.83}$$

Recall that the Green function related to the problem (3.82) is given by

$$G_f(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, 0 \le s \le t \le 1, \\ \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, 0 \le t \le s \le 1 \end{cases} \tag{3.84}$$

Consider the space $X = (C[0,1],\mathbb{R}^n)$, $E = C[0,1]$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in C[0,1]$ and define $P \in E$ by $P = \{(x,y) \in E : x(t), y(t) \ge 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$.

This space defines the metric $\rho : X \times X \to E$ such that

$$\rho(x,y) = \sup_{t\in[0,1]} \{|x(t) - y(t)|\}\psi(t) \tag{3.85}$$

$\forall x, y \in X$ and $\psi(t) = e^t$. Then (X,ρ) is a TVS valued cone metric space.

A function $x \in C([0,1],X)$ is a unique solution of the fractional differential integral equation (3.83) if and only if $x = u^*$ is a solution of the nonlinear fractional differential equation (3.82).

Now, we prove the following theorem:

Theorem 3.10 *Suppose the following conditions hold:*

(i) *$G_f(t,s) \in C([0,1]\times[0,1],X) \geq 0$ for all $t,s \in [0,1]$;*

(ii) *$\int_0^1 G_f(t,s) \leq \gamma(s)$ for all $t,s \in [0,1]$;*

(iii) *$g \in C([0,1]\times X, X)$ is sequentially continuous;*

(iv) *there exists a continuous function $g : [0,1]\times\mathbb{R} \to \mathbb{R}_+$, such that*

$$|g(t,x(s)) - g(t,y(s))| \leq e^{-\vartheta}\gamma(s)|x(s)-y(s)|,$$

for all $t \in [0,1]$ and $\vartheta > 0$, such that

$$\gamma(s) = \frac{t^{\alpha-1}[\alpha\lambda + \alpha(\alpha+1)] - (\alpha+1)[\alpha t^{\alpha} + \lambda t^{\alpha}]}{\alpha(\alpha+1)(\alpha-\lambda)\Gamma(\alpha)}.$$

Then, the fractional differential Equation 3.82 *has a common solution as a fixed point $x^* \in C([0,1],X)$.*

Proof. Let us define a map $G,f : P \to E$ by

$$Gfx(t) \;=\; \int_0^1 G_f(t,s)g(s,u(s))ds., \tag{3.86}$$

for $t \in [0,1]$, then $Gf^n x_0$ is sequentially continuous. This implies that $f \in Gf^n x_0$ and $f^n x_0$ posses a fixed point $u^* \in Gf$. To prove the existence of fixed point of Gf, we prove that Gf is sequentially continuous and contraction. To show Gf is sequentially continuous, let $Gfx \neq Gfy$, for all $x,y \in [0,1]$. By condition (iv), we have

$$\begin{aligned}
|Gfx - Gfy| &= \Big|\int_0^1 G_f(t,s)g(s,u(s))ds - \int_0^1 G_f(t,s)g(s,y(s))ds\Big|,\\
&\leq \int_0^1 G_f(t,s)\Big|g(s,x(s)) - g(s,y(s))\Big|ds,\\
&\leq \Bigg[\int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}ds\\
&\quad + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}ds\Bigg] e^{-\vartheta}\Big|x(s)-y(s)\Big|e^t,\\
&\leq \left[\frac{t^{\alpha-1}[\alpha\lambda + \alpha(\alpha+1)] - (\alpha+1)[\alpha t^{\alpha} + \lambda t^{\alpha}]}{\alpha(\alpha+1)(\alpha-\lambda)\Gamma(\alpha)}\right] e^{-\vartheta}\Big|x(s)-y(s)\Big|,
\end{aligned}$$

This implies that

$$|Gfx, Gfy| \;\leq\; e^{-\vartheta}\gamma(s)|x-y|e^t.$$

Since $\gamma(s) < 1$, we have

$$|Gfx, Gfy| \;\leq\; e^{-\vartheta}|x-y|e^t.$$

Thus for each $x, y \in X$, we have

$$\rho(Gfx, Gfy) \;\leq\; e^{-\vartheta}\mathbb{M}(x,y). \tag{3.87}$$

Taking logarithms on both sides of (3.87) using $F_1(z) = \ln(z)$ and the condition of F, we get

$$\ln(\rho(Gfx, Gfy)) \;\leq\; \ln(e^{-\vartheta}\mathbb{M}(x,y)).$$

Equivalently

$$\vartheta + F(\rho(Gfx, Gfy)) \;\leq\; F(\mathbb{M}(x,y)), \tag{3.88}$$

where

$$\mathbb{M}(x,y) \;=\; \frac{\rho(Gx, Gfx) + \rho(Gy, Gfy)}{2}. \tag{3.89}$$

Using (3.89) in (3.88) and applying F-Kannan Suzuki type conditions leads to

$$\begin{aligned} \frac{1}{2}\rho(x, Gx) \;&<\; \rho(x,y) \\ \implies \vartheta + F(\rho(Gfx, Gfy)) \;&\leq\; F\left[\frac{\rho(Gx, Gfx) + \rho(Gy, Gfy)}{2}\right], \end{aligned} \tag{3.90}$$

For $\gamma \in [0,1)$, $\vartheta > 0$ satisfies F-Kannan-Suzuki type mapping. Therefore, Tf is a contraction mapping on X. Since all the conditions of Theorem 3.10 are satisfied. Therefore, there exists $u^* \in C([0,1])$ a common fixed point of G and f, that is, u^* is a solution to fractional nonlinear differential equation (3.82). □

3.5.5 The Existence of Coincidence Solution for Non-linear Volterra-Integral Equations

This section investigates the coincidence solution for nonlinear Volterra-integral equations in the setting of TVS valued cone metric spaces. Nieto and Rodrigurz-Lopez (2005, 2007) initiated the study of the existing solution of an ordinary differential equation. Since then, several authors have utilised his ideas to find the solution of ODE. In detail, one can see the literature in Kilbas *et al.* (2006), Harjani and Sadarangani (2010), Yan *et al.* (2012), Gupta *et al.* (2017), Borisut *et al.* (2019) and the references therein.

Integral equation methods are very useful for solving many research problems in applied mathematical sciences like mathematical economics and optimal control theory because this problem is often reduced to integral equations.

Integral equations appear in several forms. However, in this section we are interested with the integral equation, namely; Volterra integral-differential equation which is of the form

$$u^n(t,x) \;=\; f(t,x)+\int_a^x K(x,t,u(t))dt, \; where \; u^n=\frac{d^nu}{dx^n}.$$

The following integral equation is inspired by Corduneanu (1991), Pachpatte (2010), Azam and Beg (2013), Abbas *et al.* (2016) and Nashine *et al.* (2011)

$$\begin{aligned} u(x,y) \;&=\; l(x,y)+\int_0^x g(x,y,\varepsilon,u(\varepsilon,y))d\varepsilon \\ &+\int_0^x\int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))d\varepsilon d\sigma, \end{aligned} \tag{3.91}$$

where l,g,h are given functions and u is unknown function to be found.

Let $C(G,f)$ be the class of continuous functions from the set G to the set f. We denote $E=\mathbb{R}^+\times\mathbb{R}^+$, $E_1=\{l(x,y,s):0\le s\le x\le\infty, y\in\mathbb{R}^+\}$ and $E_2=\{l(x,y,s,t):0\le s\le x\le\infty, 0\le t\le y\le\infty\}$. We require that $l\in C(E,\mathbb{R})$, $g\in C(E_1\times\mathbb{R},\mathbb{R})$ and $h\in C(E_2\times\mathbb{R},\mathbb{R})$

Denote by X the space of functions $z\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R})$ satisfying

$$|z(x,t)| \;=\; O(e^{\lambda(x+y)}),$$

where λ is a positive constant, that is,

$$|z(x,y)| \;\le\; M_0(e^{\lambda(x+y)}), \tag{3.92}$$

for constant $M_0>0$. Let $(X,\|.\|)$ be a Banach space. Define a norm in the space X by

$$|z|_X=\sup_{(x,y)\in E}\Big[|z(x,y)|e^{(-\lambda(x+y))}\Big]. \tag{3.93}$$

Define the mapping $G,f:X\times X\to[0,\infty)$ by

$$\begin{aligned} Gf^nu(x,y) \;&=\; l(x,y)+\int_0^x g(x,y,\varepsilon,u(\varepsilon,y))d\varepsilon \\ &+\int_0^x\int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))d\varepsilon d\sigma, \end{aligned} \tag{3.94}$$

and

$$\begin{aligned} Gf^n v(x,y) &= l(x,y) + \int_0^x g(x,y,\varepsilon,v(\varepsilon,y))d\varepsilon \\ &\quad + \int_0^x \int_0^y h(x,y,\sigma,\tau,v(\sigma,\tau))d\varepsilon d\sigma, \end{aligned} \tag{3.95}$$

for $u,v \in X$.

The coincidence fixed point of $Gf^n u$ and $Gf^n v$ is also a solution of the Volterra integral-differential equation 3.91.

Now, we prove the result by establishing the existence solution of a coincidence fixed point for a pair of self mappings:

Theorem 3.11 *Suppose the following conditions hold:*

(i) *for the continuous functions $g,h \in X$, we have*

$$|g(x,y,\varepsilon,u(\varepsilon,y)) - g(x,y,\varepsilon,v(\varepsilon,y))| \leq \gamma_1(x,y,\varepsilon)|u-v|,$$

and

$$|h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,v(\sigma,\tau))| \leq \gamma_2(x,y,\sigma,\tau)|u-v|,$$

where $\gamma_1 \in C(E_1,[0,\infty))$ and $\gamma_2 \in C(E_2,[0,\infty))$.

(ii) *there exists a non-negative constant δ such that $\delta < 1$ and*

$$\int_0^x \gamma_1(x,y,\varepsilon)e^{\lambda(\varepsilon+y)}d\varepsilon + \int_0^x \int_0^y \gamma_2(x,y,\sigma,\tau)e^{\lambda(\sigma+\tau)}d\tau d\sigma \leq \delta e^{\lambda(x+y)-\vartheta},$$

for all $x,y,\varepsilon,\sigma,\tau \in E_1 \cup E_2$;

Then, the nonlinear Volterra-integral equation (3.91) *has a unique solution in $E_1 \cup E_2$ which is the coincidence fixed point of* (3.94) *and* (3.95).

Proof. Let $G,f : X \to X$ be two operators such that $Gf^n u \in X$ and $Gf^n v \in X$. Now we verify that the two operators are contractive maps in X. Let $u,v \in X$. On contrary we claim that G and f are not contractive maps in X. From (3.94) and (3.95), using condition (i) and (ii) of Theorem 3.11 we have

$$\begin{aligned} |Gf^n u - Gf^n v|_X &= l(x,y) + \int_0^x g(x,y,\varepsilon,u(\varepsilon,y))d\varepsilon \\ &\quad + \int_0^x \int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))d\tau d\sigma \\ &\quad - l(x,y) - \int_0^x g(x,y,\varepsilon,v(\varepsilon,y))d\varepsilon \\ &\quad - \int_0^x \int_0^y h(x,y,\sigma,\tau,v(\sigma,\tau))d\tau d\sigma, \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t |g(x,y,\varepsilon,u(\varepsilon,y))-g(x,y,\varepsilon,v(\varepsilon,y))|d\varepsilon \\
&\quad +\int_0^x\int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau))-h(x,y,\sigma,\tau,v(\sigma,\tau))|d\tau d\sigma, \\
&\leq \Bigg[\int_0^x \gamma_1(x,y,\varepsilon)e^{\lambda(x+y)}d\varepsilon + \\
&\quad \int_0^x\int_0^y \gamma_2(x,y,\sigma,\tau)e^{\lambda(\sigma+\tau)}d\tau d\sigma\Bigg]|u-v|_X, \\
&\leq \delta e^{\lambda(x+y)-\vartheta}|u-v|_X, \\
&\leq \delta e^{\lambda(x+y)-\vartheta}|u-v|_X, \\
|Gf^nu-Gf^nv|_X &\leq \delta e^{-\vartheta}|u-v|_X e^{\lambda(x+y)}, \\
\rho(Gfu,Gfv) &\leq \delta e^{-\vartheta}M(u,v), \qquad (3.96)
\end{aligned}$$

which is a contradiction. Hence u is a common fixed of G and f, also a solution to integral equation (3.91).

From (3.96), since $\delta < 1$ and using $(FKS2)$ of Definition 3.8, where

$$M(u,v)=\frac{\rho(Gu,Gfu)+\rho(Gv,Gfv)}{2}, \qquad (3.97)$$

we have

$$\rho(Gfu,Gfv) \leq e^{-\vartheta}M(u,v). \qquad (3.98)$$

Using $F_1(z)=\ln z$ by taking naural logarithms in both sides of (3.98), we get

$$\vartheta+\rho(Gfu,Gfv) \leq M(u,v). \qquad (3.99)$$

By (3.97) and (3.97), we obtain an F-Kannan Suzuki contraction as defined in Definition 3.8. Thus, all conditions imposed in Theorem 3.5 and Theorem 3.11 are satified. Hence, u^* is a common fixed point of G and f in X □

CHAPTER FOUR

FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS WITH SOME APPLICATIONS

4.1 Introduction

Nadler (1969) extended the Banach principle to multivalued mappings in complete metric spaces by using Hausdorff concept of taking a measure of distance between elements of two sets. For more details of this concept, we refer the reader to subsection 2.4.1. The multi-valued mappings theory has many applications in diverse areas, such as in control theory, approximation theory, differential equations and economics.

Similarly, Ali and Kamran (2016) proved a multivalued fixed point in metric spaces by combining the concepts of α-admissible mappings, control functions (Mohammadi *et al.* 2013, Sintunavarat W 2015) and F-contractions (Wardowski 2012) to get a generalised contraction named α-F-contraction. For more details, one can refer to Abbas *et al.* (2016), Durmaz and Altun (2016), Secelean (2016) and Gopal *et al.* (2017). Later, Chifu and Petrusel (2017) proved fixed point results for multivalued Hardy Rogers contraction mapping in b-metric space.

Furthermore, Sgroi and Vetro (2013) initiated and proved some fixed point results for closed multivalued F-contractions or multivalued mappings that satisfy an F-contractive condition Hardy-Rogers-type, in the setting of complete metric spaces or complete ordered metric spaces.

Recently, Qawaqneh *et al.* (2019) presented some fixed point results by characterising a weak contractive condition based on using C-class function (Ansari 2014) and α-admissible, named; (α-F)-admissible multivalued mapping type S in the setting of b-metric spaces, where $F \in C$. Rao *et al.* (2020) presented some fixed point theorems for self mappings satisfying generalised (ϕ, ψ)-weak contraction conditions in partially ordered complete b-metric space with parameter $s \geq 1$, y generalising the results due to

Gupta and Dass (1975) and Jaggi (1977).

This chapter is divided into five sections, giving the results for fixed point theorems using multivalued mappings with their applications to integral and differential equations, by extending the concept of Banach contraction principle to multivalued notion using Nadler (1969) concept.

In section 4.2, we discuss some preliminary results; definitions, theorems and lemmas. In section 4.3, we prove a fixed point theorem for multivalued weakly α-F admissible mappings in partial metric spaces, which generalise several results in the literature, especially the main results of Theorem 4.8 by Ali and Kamran (2016).

Inspired by the work due to Sgroi and Vetro (2013), in section 4.4, we prove fixed point theorem for multivalued Hardy-Rogers F-contraction in ordered partial metric spaces by extending Theorem 4.6 by Sgroi and Vetro (2013).

Section 4.5 is motivated by the results due to Qawaqneh *et al.* (2019), Rao *et al.* (2020), Shoaib *et al.* (2020) and Hammad (2020). Using these results, we investigate the consequences of multivalued mapping using Jaggi-Hardy-Rogers type-F-$F_{\mathscr{C}}$-contraction in partially ordered b-metric space as defined in subsection 2.3.2. By extending Theorem 4.3 due to Qawaqneh *et al.* (2019) and Theorem 4.4 due to Rao *et al.* (2020). In our results, we prove the fixed point theorem using Jaggi-Hardy-Rogers type F-$F_{\mathscr{C}}$ multivalued mapping in b-metric spaces, where $\mathscr{C}$ is a C-class function (Ansari 2014) and F-contraction (Wardowski 2012). For a brief discussion on multivalued b-metric space, we refer the reader to subsection 2.4.2.

These results extend and generalise several other works in the literature. We provide examples of the use of theorems proved. Also, corollaries are introduced for extensions of the results established.

In section 4.6 covers some applications of the proved theorems in the above sections. We provide a solution of the Volterra integral equation as an application for Theorem 4.7. We also prove the existence of a solution to the first-order periodic problem and the existence solution of the ordinary differential equation using *b*-metric space concept as an application for Theorem 4.8.

4.2 Preliminaries

This subsection gives some definitions, lemma and fundamental theorems, which will be utilised to prove, extend and generalises the results.

Jaggi (1977) gave a generalisation of the Banach contraction principle by extending the results due to Dass and Gupta (1975) in metric space as follows:

Definition 4.1 *(Jaggi 1977) Let (X,d) be a complete metric space. A continuous self-mapping T on a set X is called Jaggi contraction type if*

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \beta d(x,y), \tag{4.1}$$

for all $x,y \in X$, $x \neq y$, and some $\alpha,\beta \in (0,1)$ with $\alpha+\beta < 1$.

Khan *et al.* (1984) established some fixed point theorems by altering the distance between the points and proved their results in complete and compact metric spaces.

Definition 4.2 *(Khan et al. 1984) A self map ϕ defined on $[0,+\infty)$ is said to be an altering distance function, if the following properties are satisfied:*

(i) $\phi(t) = 0 \Leftrightarrow t = 0,$

(ii) *ϕ is monotonically non-decreasing,*

(iii) *ϕ is continuous.*

Remark 4.1 *Let Φ and Ψ be define as follows:*

(1) *Let us denote the set of all altering distance function on $[0,\infty)$ by Φ .*

(2) *Similarly, Let us denote the set of all lower semi-continuous function on $[0,\infty)$ by Ψ with $\psi(t) = 0$ if and only if $t = 0$.*

Example 4.1 *(Khan et al. 1984) The following is an example which saisfies the condition of altering distance functions:*

$$\phi(t) = \begin{cases} t, & t \in [0,1], \\ \frac{1}{t}, & t \in (1,\infty). \end{cases}$$

If ϕ is a comparison function, then, it satisfies the following properties:
Definition 4.3 (Berinde 1993) Let $s \geq 1$ be real number. If $\phi : [0,+\infty) \to [0,+\infty)$ is b comparison function, then we have the following:

(1) the series $\sum_{k=0}^{\infty} s^k \phi^k$, converges for any $t \in \mathbb{R}^+$

(2) the function $S_b : [0,+\infty) \to [0,+\infty)$ defined by

$$S_b = \sum_{k=0}^{\infty} s^k \phi^k, t \in [0,+\infty),$$

is increasing and continuous at 0.

Definition 4.4 *(Samet et al. 2012) Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that T is α-admissible if $x,y \in X$, $\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$.*

Ansari (2014) gave a generalisation concept of C-class functions:
Definition 4.5 *(Ansari 2014) A mapping $F_{\mathscr{C}} : [0,\infty) \times [0,+\infty) \to \mathbb{R}$ is said to be a C-class function if it continuous and for $s,t \in [0,\infty)$, $F_{\mathscr{C}}$ satisfies the following properties:*

(i) $F_{\mathscr{C}}(r,t) \leq r$;

(ii) $F_{\mathscr{C}}(r,t) = 0 \implies$ *that either* $r = 0$ *or* $t = 0$.

We denote the set of C-class functions by $\mathscr{C}$. The following are examples which satisfies $F_{\mathscr{C}}$ functions:
Example 4.2 *(Ansari 2014, Ansari and Kaewcharoen 2016) The following functions $F_{\mathscr{C}} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are elements of, $\mathscr{C}$ for all $s,t \in \mathbb{R}^+$;*

(1) $F_{\mathscr{C}}(s,t) = s - t$, $F_{\mathscr{C}}(s,t) = s \Rightarrow t = 0$;

(2) $F_{\mathscr{C}}(s,t) = ms, 0 < m < 1, F_{\mathscr{C}}(s,t) = s \Rightarrow 0$;

(3) $F_{\mathscr{C}}(s,t) = s\beta(s), \beta : \mathbb{R}^+ \to [0,1)$, *and is continuous,* $F_{\mathscr{C}}(s,t) = s \Rightarrow 0$;

(5) $F_{\mathscr{C}}(s,t) = s - \psi(s), F_{\mathscr{C}}(s,t) = s \Rightarrow 0$, *where* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function such that* $\psi(t) = 0 \Leftrightarrow t = 0$;

(6) $F_{\mathscr{C}}(s,t) = \phi(s), F_{\mathscr{C}}(s,t) = s \Rightarrow 0$, *where* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function such that* $\phi(0) = 0$, *and* $\phi(t) > 0$ *for* $t > 0$.

Definition 4.6 *(Ali and Kamran 2016) Let f be a self mapping on a non empty set X and $\alpha : X \times X \longrightarrow [0,\infty)$ be a mapping. We say that T is weakly α-admissible if the following conditions hold:*

(i) for $x \in X$ with $\alpha(x, Tx) \geq 1$,

(ii) there exists Tx for $x \in X$ with $\alpha(Tx, TTx) \geq 1$.

From Ali and Kamran (2016), we get the following definition of an α-F-weakly contraction mapping as follows:

Definition 4.7 *(Ali and Kamran 2016)* Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be function. A mapping $T : X \to CB(X)$ is α-F-contraction if there exists a continuous function F in $\mathscr{F}$ and $\tau > 0$ such that

$$\tau + F(\alpha(x,y)H(Tx,Ty)) \quad \leq \quad F(\mathbb{M}(x,y)),$$

for each $x, y \in X$, whenever $\min\{\alpha(x,y)H(Tx,Ty), \mathbb{M}(x,y)\} > 0$, where

$$\mathbb{M}(x,y) \quad = \quad \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\} + Ld(y,Tx)$$

and $L \geq 0$.

We consider the following theorem by Ali and Kamran (2016):

Theorem 4.1 *(Ali and Kamran 2016) Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be an α-F-contraction satisfying the following conditions:*

(i) T is strictly α-admissible mapping;

(ii) there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then T has a fixed point.

In this study, we use the following observation from Cosentino *et al.* (2015).

Remark 4.2 *(Cosentino* et al. *2015)* Given sets A, B. For every $\tau > 0$ there is $h > 1$ such that

$$F\big(hsH_b(A,B)\big) < F\big(sH_b(A,B)\big) + \tau.$$

Definition 4.8 *(Cosentino et al. 2015) Let (X,d,s) be a b-metric space. A multivalued mapping $T : X \to CB(X)$ is called an F-contraction of Nadler type if there exists $F \in \mathscr{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$2\tau + F(sH_b(Tx,Ty)) \leq F(d(x,y)) \tag{4.2}$$

for all $x, y \in X$ with $Tx \neq Ty$.

We recall the definition for the relation of two maps in partially orderd metric space from Beg and Butt (2010) as follows:

Definition 4.9 *(Beg and Butt 2010) Let A and B be two non-empty subsets of* $(X, \preceq)$*, the relation between A and B are denoted and defined as follows:*

(1) $A \prec_1 B$*: if for every* $a \in A$ *there exists* $b \in B$ *such that* $a \preceq b$*,*

(2) $A \prec_2 B$*: if for every* $b \in B$ *there exists* $a \in A$ *we have* $a \preceq b$*,*

(3) $A \prec_3 B$*: if* $A \prec_1 B$ *and* $A \prec_2 B$*.*

Kumar (2019) extended the results due to Durmaz *et al.* (2016), where he introduced the following definition and theorem on ordered partial metric spaces using two compatible mappings:

Definition 4.10 *(Kumar 2019) Let*$(X, \preceq, p)$ *be an ordered partial metric space and* $T : X \to X$ *be a mapping. Also let* $Y = \{(x,y) \in X \times X : x \preceq y, p(Tx,Ty) > 0\}$*. We say that T is an ordered F-contraction if* $F \in \mathscr{F}$ *and there exists* $\tau > 0$ *such that for all* $(x,y) \in Y$*, we have*

$$\tau + F(p(Tx,Ty)) \leq F(p(x,y)). \tag{4.3}$$

Theorem 4.2 *(Kumar 2019) Let* $(X, \preceq)$ *be partial ordered set and suppose that there exists a partial metric space on X such that* (X, p) *is a complete partial metric space. Suppose T and g are continuous self F-contraction mappings on X,* $T(X) \subseteq g(X)$*, T is monotone g-non decreasing mapping and*

$$\tau + F(p(Tx,Ty)) \leq F(\mathbb{M}(x,y)),$$

where

$$\mathbb{M}(x,y) = \max\left\{ p(gx,gy), p(gx,Tx), p(gy,Ty), \frac{1}{2}[p(gx,Ty) + p(gy,Tx)] \right\},$$

for all $x,y \in X$ *for which gx and gy are comparable and* $\tau > 0$*. If there exists* $x_0 \in X$ *such that* $gx_0 \preceq Tx_0$ *and T and g are compatible, then T and g have a coincident point.*

The ideals of the following definition is taken from Sgroi and Vetro (2013), and Cosentino *et al.* (2015).

Definition 4.11 *Let* (X, d, s) *be a b-metric space. A multivalued mapping* $T : X \to CB(X)$ *is called an F-contraction of Hardy-Rogers type if there exists* $F \in \mathscr{F}$ *and* $\tau \in \mathbb{R}^+$ *such that*

$$\begin{aligned} 2\tau + F\big(sH_b(Tx,Ty)\big) \quad \leq \quad & F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \\ & \delta d(x,Ty) + Ld(y,Tx)), \end{aligned}$$

for all comparable $x, y \in X$ *with* $Tx \neq Ty$*, where* $\alpha, \beta, \gamma, \delta, L \geq 0$*,* $\alpha + \beta + \gamma + 2\delta = 1$ *and* $\gamma \neq 1$

Qawaqneh *et al.* (2019) proved the following theorem by considering F as a C-class function.

Theorem 4.3 *(Qawaqneh et al. 2019)* Let (X,d,s) be a complete b- metric space with a constant $s \geq 1$ and $T : X \to CB(X)$ be an α-F-Multivalued mapping. Assume that there exists $\alpha : X \to [0,\infty)$, $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\alpha(\theta_0,\theta_1)H_b(f\theta_0,f\theta_1) \leq F(\psi(M(\theta_0,\theta_1)),\phi(M(\theta_0,\theta_1))),$$

where

$$\begin{aligned} M(\theta_0,\theta_1) &= \Bigg\{ d_b(\theta_0,\theta_1), \frac{d_b(\theta_0,f\theta_0)}{1+d_b(\theta_0,f\theta_0)}, \frac{d_b(\theta_1,f\theta_1)}{1+d_b(\theta_1,f\theta_1)}, \\ &\qquad \frac{1}{2}[d_b(\theta_0,f\theta_1)+d_b(\theta_1,f\theta_0)] \Bigg\}, \end{aligned}$$

for all $\theta_0,\theta_1 \in X$ assume that:

(i) f is α-admissible of type s and ψ is continuous;

(ii) there exists $\theta_0 \in X$ and $x_1 \in T\theta_0$ with $\alpha(\theta_0,\theta_1) \geq s$;

(iii) $\{\theta_n\}$ a sequence in X with $\theta_n \to \theta$ as $n \to \infty$ and $\alpha(\theta_n,\theta_{n+1}) \geq s \ \forall\, n \in \mathbb{N} \cup \{0\}$, then $\alpha(\theta_n,\theta) \geq s$ forall $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point.

Rao *et al.* (2020) proved the following theorem in partially ordered b-metric spaces with parameter $s \geq 1$.

Theorem 4.4 *(Rao et al. 2020)* Let $(X,d,s,\leq)$ be a complete partially ordered b-metric space with parameter $s \geq 1$. Let $T : X \to X$ be continuous non decreasing mappings with regard to $\leq$ such that there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Suppose that

$$\phi(sd(Tx,Ty)) \leq \phi(M(x,y)) - \psi(M(x,y)),$$

where $\phi \in \Phi, \psi \in \Psi$, for any $x,y \in X$ with $x \leq y$, and

$$M(x,y) = \left\{ \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \frac{d(x,Ty)[1+d(y,Tx)]}{1+d(x,y)}, \beta d(x,y) \right\},$$

Then T has a fixed point in X.

4.3 Existence Results for Multi-valued α-F Contraction Mappings in Partial Metric Spaces

We start our first result by slightly modifying the Definitions 4.4 given in (Samet *et al.* 2012).

Definition 4.12 *Let* $\alpha : X \times X \to [0,\infty)$ *be a function in a partial metric space* (X,p). *A mapping* $T : X \to CB^p(X)$ *is said to be strictly* α*-admissible if for each* $x \in X$ *and* $y \in Tx$ *such that* $\alpha(x,y) > 1$ *we have* $\alpha(y,z) > 1$ *for each* $z \in Ty$.

In order to develop our main result, we modify Definition 4.7 as follows:

Definition 4.13 *Let* (X,p) *be a partial metric space and* $\alpha : X \times X \to [0,\infty)$ *be a function. A mapping* $T : X \to CB^p(X)$ *is an* α*-*F*-contraction if there exists a continuous function* F *in* $\mathscr{F}$ *and* $\tau > 0$ *such that*

$$\tau + F(\alpha(x,y)H_p(Tx,Ty)) \leq F(\mathbb{M}(x,y)), \tag{4.4}$$

for each $x,y \in X$, *whenever* $\min\{\alpha(x,y)H_p(Tx,Ty), \mathbb{M}(x,y)\} > 0$ *and* $q,r \geq 0$,

where

$$\mathbb{M}(x,y) = \max\left\{p(x,y), \frac{p(x,Tx)+p(y,Ty)}{q}, \frac{p(x,Ty)+p(y,Tx)}{r}\right\}.$$

By extending Theorem 4.1, we prove following result:

Theorem 4.5 *Let* (X,p) *be a complete partial metric space, and* $T : X \to CB^p(X)$ *be an* α*-*F*-contraction satisfying the following conditions:*

(i) *T is strictly* α*-admissible mapping;*

(ii) *there exists* $x_0 \in X$ *and* $x_1 \in Tx_0$ *with* $\alpha(x_0,x_1) > 1$;

(iii) *for any sequence* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n,x_{n+1}) > 1$ *for each* $n \in \mathbb{N}$, *we have* $\alpha(x_n,x) > 1$ *for each* $n \in \mathbb{N}$.

Then there exists $x^* \in X$ *such that* $Tx^* = x^*$ *and* $p(x^*,x^*) = 0$. x^* *is a fixed point of* T.

Proof. Let $x_0 \in X$ be an arbitrary point and choose $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) > 1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and the proof is completed.

If however $x_1 \notin Tx_1$, then apply (4.4) with $x = x_0$ and $y = x_1$ as follows:

$$\tau + F\big(\alpha(x_0,x_1)H_p(Tx_0,Tx_1)\big) \leq F[\max\{\mathbb{M}(x_0,x_1)\}], \tag{4.5}$$

where

$$\begin{aligned}
&\mathbb{M}(x_0,x_1)\\
&= \max\left\{p(x_0,x_1), \frac{p(x_0,Tx_0)+p(x_1,Tx_1)}{q}, \frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{r}\right\}\\
&\leq \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,Tx_1)}{q}, \frac{p(x_0,Tx_1)+p(x_1,x_1)}{r}\right\}
\end{aligned}$$

because $x_1 \in Tx_0, x_2 \in Tx_1$, we have

$$\leq \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,x_2)}{q}, \frac{p(x_0,x_2)+p(x_1,x_1)}{r}\right\},$$

by $(P4)$ of Definition 2.10, we have

$$\begin{aligned}
\leq \;\; & \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,Tx_1)}{q},\right.\\
& \left.\frac{p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1)+p(x_1,x_1)}{r}\right\},
\end{aligned}$$

using $(P1)$ and (2.20) in above inequality, we get

$$\begin{aligned}
&\leq \;\; \max\left\{p(x_0,x_1), \frac{p(x_0,x_1)+p(x_1,x_2)}{q},\right.\\
&\qquad \left.\frac{p(x_0,x_1)+p(x_1,x_2)}{r}\right\},\\
\Rightarrow \mathbb{M}(x_0,x_1) &\leq \;\; p(x_0,x_1). \qquad (4.6)
\end{aligned}$$

We substitute (4.6) into (4.5) and get

$$\tau + F\big(\alpha(x_0,x_1)H_p(Tx_0,Tx_1)\big) \leq F\big(p(x_0,x_1)\big). \qquad (4.7)$$

As $\alpha(x_0,x_1) > 1$, by Lemma 2.17 there exists $x_2 \in Tx_1$ such that

$$p(x_1,x_2) \leq \alpha(x_0,x_1)H_p(Tx_0,Tx_1). \qquad (4.8)$$

As F is an increasing function we have

$$F\big(p(x_1,x_2)\big) \leq F\big(\alpha(x_0,x_1)H_p(Tx_0,Tx_1)\big). \qquad (4.9)$$

Inserting (4.8) in (4.7) we get

$$\tau + F\big(p(x_1,x_2)\big) \leq F\big(p(x_0,x_1)\big). \qquad (4.10)$$

Since T is strictly α- admissible, according to Definition 4.12, we have $\alpha(x_0,x_1) > 1 \Rightarrow \alpha(x_1,x_2) > 1$. If $x_2 \in Tx_2$, then x_2 is a fixed point and the proof is completed. Suppose $x_2 \notin Tx_2$. We apply Equation 4.4 with $x = x_1, y = x_2$ and get

$$\tau + F\big(\alpha(x_1,x_2)H_p(Tx_1,Tx_2)\big) \leq F[\max\{\mathbb{M}(x_1,x_2)\}], \qquad (4.11)$$

where

$$\begin{aligned}&\mathbb{M}(x_1,x_2)\\&\quad=\max\left\{p(x_1,x_2),\frac{p(x_1,Tx_1)+p(x_2,Tx_2)}{q},\frac{p(x_1,Tx_2)+p(x_2,Tx_1)}{r}\right\}\\&\Rightarrow\mathbb{M}(x_1,x_2)\leq p(x_1,x_2).\end{aligned}\tag{4.12}$$

On applying (4.12) to (4.11) we get

$$\tau+F\big(\alpha(x_1,x_2)H_p(Tx_1,Tx_2)\big)\leq F\big(p(x_1,x_2)\big).\tag{4.13}$$

As $\alpha(x_1,x_2)>1$, by Lemma (2.17) there exists $x_3\in Tx_2$ such that

$$p(x_2,x_3)\leq\alpha(x_1,x_2)H_p(Tx_1,Tx_2).\tag{4.14}$$

F is an increasing function, therefore

$$F\big(p(x_2,x_3)\big)\leq F\big(\alpha(x_1,x_2)H_p(Tx_1,Tx_2)\big).\tag{4.15}$$

On applying (4.15) to (4.13), we get

$$\tau+F(p(x_2,x_3))\leq F\big(p(x_1,x_2)\big).\tag{4.16}$$

Therefore (4.16) becomes

$$\begin{aligned}&\tau+F(p(x_2,x_3))\leq F(p(x_1,x_2))\\&\Rightarrow F(p(x_2,x_3))\leq F(p(x_1,x_2))-\tau\\&\Rightarrow F(p(x_2,x_3))\leq F(p(x_0,x_1))-2\tau,\text{ by (4.10)}.\end{aligned}\tag{4.17}$$

Continuing in the same manner, we form a sequence $\{x_n\}$ which reaches one of the following scenarios. Either $x_n\in Tx_n$ for some $n\in\mathbb{N}$. In this case, x_n is the fixed point and the proof is completed.

Otherwise, we have for all $n\in\mathbb{N}$, $x_n\notin Tx_n, x_n\in Tx_{n-1}$,
$\alpha(x_{n-1},x_n)>1$ and

$$F\big(p(x_n,x_{n+1})\big)\leq F\big(p(x_0,x_1)\big)-n\tau.\tag{4.18}$$

We determine the limit $n\to\infty$ of (4.18) and get

$$\lim_{n\to\infty}F\big(p(x_n,x_{n+1})\big)=-\infty.$$

By condition (F_2), this implies

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0.\tag{4.19}$$

Let $\alpha_n = p(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. By condition $(F3)$, there exist $k \in (0,1)$ and such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0.$$

From (4.17) we have

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq -n\alpha_n^k \tau < 0 \text{ for each } n \in \mathbb{N}. \tag{4.20}$$

Letting $n \to \infty$ in (4.20) we get

$$\lim_{n\to\infty} n\alpha_n^k = 0. \tag{4.21}$$

This implies there exists $n_1 \in \mathbb{N}$ such that $n\alpha_n^k < 1$ for all $n > n_1$. Therefore we have

$$\alpha_n < \frac{1}{n^{1/k}}. \tag{4.22}$$

We now show that $\{x_n\}$ is a p-Cauchy sequence. Consider $m, n \in \mathbb{N}, m < n < n_1$. By $(P4)$ of Definition 2.10, we have

$$\begin{aligned} p(x_m, x_n) &\leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) - \sum_{i=m+1}^{n-1} p(x_i, x_i) \\ &\leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{\infty} p(x_i, x_{i+1}) \\ &= \sum_{i=m}^{\infty} \alpha_i \\ &\leq \sum_{i=m}^{\infty} \frac{1}{i^{1/k}}, \text{ from (4.22).} \end{aligned}$$

The series $\sum_{i=m}^{\infty} \dfrac{1}{i^{1/k}}$ converges as it is a p-series with an exponent greater than one. This implies $\lim_{m,n\to\infty} p(x_m, x_n) = 0$. This makes $\{x_n\}$ a Cauchy sequence by (iii) of Definition 2.11.

As (X, p) is complete, there exists $x^\star \in X$ such that $x_n \to x^\star$. By (4.19), this means $p(x^\star, x^\star) = 0$. Also by condition (iii) of Theorem (4.5), we have $\alpha(x_n, x^\star) > 1$ for all $n \in \mathbb{N}$.

We claim that $x^\star$ is a fixed point of T, that is $p(x^\star, Tx^\star) = p(x^\star, x^\star) = 0$. Suppose $p(x^\star, Tx^\star) > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $p(x_n, Tx^\star) > 0$ for all $n > n_0$. By definition (4.13), for all $n > n_0$ and $q, r \geq 0$, we have

$$\begin{aligned}
&\tau + F\big(p(x_{n+1}, Tx^\star)\big)\\
&\quad \leq \tau + \alpha(x_n, x^\star) F\left(H_p(Tx_n, Tx^\star)\right)\\
&\quad \leq F\left(\max\left\{p(x_n, x^\star), \frac{p(x_n, Tx_n) + p(x^\star, Tx^\star)}{q}, \frac{p(x_n, Tx^\star) + p(x^\star, Tx_n)}{r}\right\}\right)
\end{aligned} \tag{4.23}$$

We let $n \to \infty$ in (4.23) and get

$$\tau + F\big(p(x^\star, Tx^\star)\big) \leq F\big(p(x^\star, x^\star)\big). \tag{4.24}$$

since $\tau > 0$, the above inequality yield a contradiction. Hence $p(x^\star, Tx^\star) = 0$. $\square$

Example 4.3 *Consider the partial metric space (X, p) where $X = \{0, 1, 2, \ldots\}$ and $p(x, y) = |x - y| + \max\{x, y\}$ for all $x, y \in X$. Define the multivalued function $T : X \to CB^p(X)$ as*

$$Tx = \begin{cases} \{0, 1\} & \text{for } 0 \leq x \leq 1;\\ \\ \{x - 1, x\} & \text{for } x > 1. \end{cases}$$

Let $\alpha : X \times X \to [0, \infty)$ be defined as

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in \{0, 1\};\\ \frac{1}{2} & \text{if } x, y > 1;\\ 0 & \text{otherwise.} \end{cases}$$

Now, we show that T is strictly α-admissible with the following cases:

Case 1 *Assume that $x = x_0$ and $y = x_1$. Let $x_0 = 0$ and $x_1 = 1$, then $x_1 \in Tx_0 = \{0, 1\}$, such that $\alpha(x_0, x_1) > 1$. Also we choose x_2 such that $x_2 \in Tx_1$, $x_2 = 0 \in Tx_1 = \{0, 1\}$, thus $\alpha(x_1, x_2) > 1$.*

Case 2 *We define $F(x) = x + \ln(x), x \in (0, \infty)$. Under this F, the Equation (4.4) simplifies to*

$$\frac{\alpha(x, y) H_p(Tx, Ty)}{\mathbb{M}(x, y)} e^{\alpha(x, y) H_p(Tx, Ty) - \mathbb{M}(x, y))} \leq e^{-\tau}. \tag{4.25}$$

We now calculate $H_p(Tx, Ty)$ for $x > y > 1$ and $q, r \geq 2$.

$$\begin{aligned}
Tx = \{x - 1, x\}, &\quad Ty = \{y - 1, y\};\\
p(x - 1, y - 1) = 2x - y - 1, &\quad p(x - 1, y) = 2x - y - 2\\
p(x, y - 1) = 2x - y + 1, &\quad p(x, y) = 2x - y.
\end{aligned}$$

$$\begin{aligned}
p(x - 1, Ty) &= \min\{p(x - 1, a), a \in Ty\}\\
&= \min\{p(x - 1, y - 1), p(x - 1, y)\}\\
&= \min\{2x - y - 1, 2x - y - 2\}\\
&= 2x - y - 2.
\end{aligned}$$

In the same manner we get

$$p(x,Ty) = 2x - y,\ p(y-1,Tx) = 2x - y - 1,\ p(y,Tx) = 2x - y - 2.$$

$$\begin{aligned}\delta_p(Tx,Ty) &= \max\{p(a,Ty), a \in Tx\}\\ &= \max\{p(x-1,Ty), p(x,Ty)\}\\ &= \max\{2x-y-2, 2x-y\}\\ &= 2x-y.\end{aligned}$$

Similarly

$$\delta_p(Ty,Tx) = 2x - y - 1.$$

Hence

$$\begin{aligned}H_p(Tx,Ty) &= \max\{\delta_p(Tx,Ty), \delta_p(Ty,Tx)\}\\ &= \max\{2x-y, 2x-y-1\}\\ &= 2x-y.\end{aligned}$$

We note that for $x > y > 1$, $\min\{\alpha(x,y)H(Tx,Ty), \mathbb{M}(x,y)\} > 0$ *and* $\mathbb{M}(x,y) \geq p(x,y) = 2x - y$. *Hence (4.25) becomes*

$$\begin{aligned}\frac{1}{2}\cdot\frac{2x-y}{\mathbb{M}(x,y)}e^{\frac{1}{2}(2x-y)-\mathbb{M}(x,y)} &\leq \frac{1}{2}\cdot\frac{2x-y}{2x-y}e^{\frac{1}{2}(2x-y)-(2x-y)}\\ &\leq \frac{1}{2}e^{-3/2},\ \textit{because}\ x > y > 1,\\ &\leq e^{-\tau}\ \textit{for}\ \tau \geq \frac{3}{2}.\end{aligned}$$

for $\tau \geq \frac{3}{2}$.

This shows that T is a multivalued α-F-contraction with contractive factor $\tau = \frac{3}{2}$ and $F(a) = \ln a + a$. For $x_0 = 0$ and $x_1 \in Tx_0 = \{0,1\}$, we obtain $\alpha(0,1) > 1$. Furthermore, we see that T is strictly α- admissible map and for any sequences $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$. Therefore, by Theorem 4.5, T has a fixed point in X.

4.4 Fixed Point Theorems for Multi-valued F-Contraction Mappings in Ordered Partial Metric Spaces

The following theorem for multivalued mapping in ordered metric spaces is due to Sgroi and Vetro (2013).

Theorem 4.6 *(Sgroi and Vetro 2013) Let* $(X,d,\preceq)$ *be an ordered complete metric space and Let* $T : X \to CB(X)$. *Assume that there exists* $F \in \mathscr{F}$ *and* $\tau \in \mathbb{R}_+$ *such that*

$$\begin{aligned} 2\tau + F\big(H(Tx,Ty)\big) \;&\leq\; F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \\ &\qquad \delta d(x,Ty) + Ld(y,Tx), \end{aligned}$$

for all comparable $x,y \in X$ *with* $Tx \neq Ty$, *where* $\alpha,\beta,\gamma,\delta,L \geq 0$, $\alpha+\beta+\gamma+2\delta = 1$ *and* $\gamma \neq 1$. *If the following condition are satisfied:*

(i) *there exists* $x_0 \in X$ *such that* $\{x_0\} \prec_1 Tx_0$;

(ii) *for* $x,y \in X$, $x \preceq y$ *implies* $Tx \prec_2 Ty$;

(iii) X *is regular.*

Then T *has a fixed point.*

We give the extended version of Definitions 4.8, 4.10 and 4.14 to an ordered multi-valued Hardy-Rogers F-contraction in partial metric spaces as follows:

Definition 4.14 *Let* $(X,\preceq,p)$ *be an ordered partial metric space and* $T : X \to CB^p(X)$ *be a multi-valued mapping. We say that* T *is an ordered multi-valued Hardy-Rogers* F*-contraction if* $F \in \mathfrak{F}$ *and there exists* $\tau > 0$ *such that for all* $x,y \in X$, *we have*

$$2\tau + F\big(H_p(Tx,Ty)\big) \leq F\big(\mathbb{M}(x,y)\big), \tag{4.26}$$

where

$$\mathbb{M}(x,y) = \alpha p(x,y) + \beta p(x,Tx) + \gamma p(y,Ty) + \delta p(x,Ty) + Lp(y,Tx)$$

for $x \preceq y \Leftrightarrow Tx \preceq Ty$, $\alpha,\beta,\gamma,\delta,L \geq 0$, $\alpha+\beta+\gamma+\delta = 1$ *and* $\gamma \neq 1$.

By extending Theorem 4.6, we prove the following theorem:

Theorem 4.7 *Let* $(X,\preceq)$ *be a partial ordered set and suppose that there exists a partial metric* p *such that* (X,p) *is a complete partial metric space. Let* $T : X \to CB^p(X)$ *be a multi-valued map. Assume that there exists* $F \in \mathfrak{F}$ *and* $\tau \in \mathbb{R}_+$ *such that* T *is a multi-valued Hardy-Rogers-type* F*-contraction which satisfy the following conditions:*

(i) *there exists* $x_0 \in X$ *such that* $x_0 \prec_1 Tx_0$;

(ii) *for* $x,y \in X$, $x \preceq y \implies Tx \prec_2 Ty$;

(iii) *if* $x_n \to x$ *is a non decreasing sequence in* X, *for all* n *and*

$$2\tau + F(H_p(Tx,Ty)) \leq F(\mathbb{M}(x,y)), \tag{4.27}$$

where

$$\mathbb{M}(x,y) = \alpha p(x,y) + \beta p(x,Tx) + \gamma p(y,Ty) + \delta p(x,Ty) + Lp(y,Tx)$$

for $x,y \in X$, $\tau > 0$, $\alpha,\beta,\gamma,\delta,L \geq 0$, $\alpha+\beta+\gamma+\delta = 1$ *and* $\gamma \neq 1$. *Then* T *has a fixed point.*

Proof. From assumption (i), there exists $x_0 \in X$ such that $x_0 \prec_1 Tx_0$. Choosing $x_1 \in Tx_0$, by (ii) we have $x_0 \preceq x_1 \Rightarrow Tx_0 \prec_2 Tx_1$. If $x_1 \in Tx_1$ then x_1 is a fixed point of T and we have completed our proof.

Suppose $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F(hH_p(Tx_0,Tx_1)) < F(H_p(Tx_0,Tx_1)) + \tau.$$

Now, from $F(p(x_1,Tx_1)) < F(H_p(Tx_0,Tx_1))$ and $Tx_0 \prec_2 Tx_1$, by this case we choose $x_2 \in Tx_1$ such that $F(p(x_1,x_2)) \leq F(H_p(Tx_0,Tx_1))$ and by using Lemma 2.17 as a result, we get

$$p(x_1,x_2) \leq hH_p(Tx_0,Tx_1) < H_p(Tx_0,Tx_1) + \tau,$$

$$F(p(x_1,x_2)) \leq F(hH_p(Tx_0,Tx_1)) < F(H_p(Tx_0,Tx_1)) + \tau,$$

we apply (4.27) with $x = x_0, y = x_1$ to get

$$\begin{aligned} 2\tau + F(p(x_1,x_2)) &\leq 2\tau + F(H_p(Tx_0,Tx_1)) + \tau, \\ &\leq F\big(\mathbb{M}(x_0,x_1)\big) + \tau, \\ &= F\big(\alpha p(x_0,x_1) + \beta p(x_0,Tx_0) + \gamma p(x_1,Tx_1), \\ &\quad +\delta p(x_0,Tx_1) + Lp(x_1,Tx_0)\big) + \tau, \end{aligned}$$

from $x_1 \in Tx_0, x_2 \in Tx_1$, we have

$$\begin{aligned} &\leq F\big(\alpha p(x_0,x_1), +\beta p(x_0,x_1) + \gamma p(x_1,x_2), \\ &\quad +\delta p(x_0,x_2) + Lp(x_0,x_1)\big) + \tau, \end{aligned}$$

by $(P4)$ of Definition 2.10, we get

$$\begin{aligned} &\leq F\big(\alpha p(x_0,x_1) + \beta p(x_0,x_1) + \gamma p(x_1,x_2), \\ &\quad +\delta p(x_0,x_1) + \delta p(x_1,x_2) - \delta p(x_1,x_1) + Lp(x_1,x_1)\big) + \tau \\ &= F\big((\alpha+\beta+\delta+L)p(x_0,x_1) + (\gamma+\delta)p(x_1,x_2)\big) \\ &\quad +\tau, \end{aligned}$$

using $(P1)$ and (2.20), we get

$$\begin{aligned} &\leq F\big(\alpha p(x_0,x_1) + \beta p(x_0,x_1) + \gamma p(x_1,x_2), \\ &\quad +\delta p(x_0,x_1) + \delta p(x_1,x_2)\big) + \tau \\ &= F\big((\alpha+\beta+\delta)p(x_0,x_1) + (\gamma+\delta)p(x_1,x_2)\big) \\ &\quad +\tau. \end{aligned}$$

$$\tau+F(p(x_1,x_2)) \leq F\big((\alpha+\beta+\delta)p(x_0,x_1)+(\gamma+\delta)p(x_1,x_2)\big). \tag{4.28}$$

As F is an increasing function, by (F_1) (4.28) implies

$$\begin{aligned}&\Rightarrow F(p(x_1,x_2)) < F\big((\alpha+\beta+\delta)p(x_0,x_1)+(\gamma+\delta)p(x_1,x_2)\big)\\ &\Rightarrow p(x_1,x_2) < (\alpha+\beta+\delta)p(x_0,x_1)+(\gamma+\delta)p(x_1,x_2)\\ &\Rightarrow (1-\gamma-\delta)p(x_1,x_2) < (\alpha+\beta+\delta)p(x_0,x_1).\end{aligned} \tag{4.29}$$

From the assumption we have

$$\alpha+\beta+\gamma+\delta=1, L=0 \text{ implying } 1-\gamma-\delta=\alpha+\beta+\delta.$$

Hence, (4.29) implies

$$p(x_1,x_2) < p(x_0,x_1). \tag{4.30}$$

Using (4.30) in (4.28) we get

$$F\big(p(x_1,x_2)\big) \leq F\big(p(x_0,x_1)\big)-\tau. \tag{4.31}$$

If $x_2 \in Tx_2$ then x_2 is a fixed point of T and the proof is completed. However, suppose $x_2 \notin Tx_2$. As $Tx_0 \prec_2 Tx_1$, $x_1 \in Tx_0$ and $x_2 \in Tx_1$, we have $x_1 \preceq x_2 \Rightarrow Tx_1 \prec_2 Tx_2$. Let us choose $x_3 \in Tx_2$. Therefore, by Lemma 2.17, we get

$$p(x_2,x_3) \leq hH_p(Tx_1,Tx_2) < H_p(Tx_1,Tx_2)+\tau.$$

$$F(p(x_2,x_3)) \leq F(hH_p(Tx_1,Tx_2)) < F(H_p(Tx_1,Tx_2))+\tau.$$

We apply (4.26) with $x=x_1, y=x_2$, we get

$$\begin{aligned}2\tau+F(p(x_2,x_3)) &\leq 2\tau+F(H_p(Tx_1,Tx_2))+\tau,\\ &\leq F\big(\mathbb{M}(x_1,x_2)\big)+\tau,\\ &= F\big(\alpha p(x_1,x_2),+\beta p(x_1,Tx_1)+\gamma p(x_2,Tx_2),\\ &\quad +\delta p(x_1,Tx_2)+Lp(x_2,Tx_1)\big)+\tau,\end{aligned}$$

Similarly, one obtains,

$$\tau+F\big(p(x_2,x_3)\big) \leq \big((\alpha+\beta+\delta)p(x_1,x_2)+(\gamma+\delta)p(x_2,x_3)\big)$$

As F is an increasing function, by (F_1), we get

$$p(x_2,x_3) < p(x_1,x_2). \tag{4.32}$$

Using (4.32) in (4.28) and (4.31) we get

$$F\big(p(x_2,x_3)\big) \leq F\big(p(x_1,x_2)\big)-\tau \leq F\big(p(x_0,x_1)\big)-2\tau. \tag{4.33}$$

Continuing in the same manner we get the sequence $\{x_n\}$ with $x_1 \prec x_2 \prec x_3 \ldots$. If $x_n \in Tx_n$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the proof is completed. Suppose $x_n \notin Tx_n$ for all $n \in \mathbb{N}$. In this case we have

$$F\big(p(x_n,x_{n+1})\big) \leq F\big(p(x_0,x_1)\big) - n\tau. \tag{4.34}$$

We notice that (4.34) is identical to (4.18). Next, proceeding as in the proof of Theorem 4.5, we obtained that $\{x_n\}$ is Cauchy sequence. Also, since (X,p) is a complete partial metric space, there is $x^\star \in X$ such that $x_n \to x^\star$, and $p(x^\star,x^\star) = 0$.

We claim that $x^\star$ is a fixed point of T. We do this by showing that $p(x^\star,Tx^\star) = p(x^\star,x^\star) = 0$. Suppose $p(x^\star,Tx^\star) > 0$. Then there exists some $n_0 \in \mathbb{N}$ such that $p(x_n,Tx^\star) > 0$ for all $n > n_0$. By (4.27) we have

$$\begin{aligned}
2\tau + F\big(p(x_{n+1},Tx^\star)\big) &\leq 2\tau + F\big(H_p(Tx_n,Tx^\star)\big) + \tau \\
&\leq F\big(\mathbb{M}(x_n,x^\star)\big) \\
&= F\big(\alpha p(x_n,x^\star) + \beta p(x_n,Tx_n) \\
&\quad + \gamma p(x^\star,Tx^\star) + \delta\big(p(x_n,Tx^\star) + p(x^\star,Tx_n)\big) + Lp(x^*,Tx_n) + \tau.
\end{aligned} \tag{4.35}$$

Taking $n \to \infty$ in (4.35) and applying the fact that F is an increasing function, we get

$$\begin{aligned}
2\tau + F\big(p(x^\star,Tx^\star)\big)& \\
&\leq F\big(\alpha p(x^\star,x^\star) + \beta p(x^\star,Tx^\star) + \gamma p(x^\star,Tx^\star) + 2\delta p(x^\star,Tx^\star) \\
&\quad + Lp(x^*,Tx^*)\big) + \tau, \\
&\leq F\big((\alpha+\beta+\gamma+\delta+L)p(x^\star,Tx^\star)\big) + \tau, \\
2\tau + F\big(p(x^\star,Tx^\star) &\leq F\big(p(x^\star,Tx^\star)\big) + \tau, \\
2\tau + F\big(p(x^\star,Tx^\star) &\leq F\big(p(x^\star,Tx^\star)\big) + \tau, \\
F\big(p(x^\star,Tx^\star) &\leq F\big(p(x^\star,Tx^\star)\big) - \tau.
\end{aligned}$$

Since $\tau > 0$, the above inequality yields a contradiction. Hence $p(x^\star,Tx^\star) = 0$ making $x^\star$ a fixed point of T. The proof is completed. □

Now, we give an example to illustrate the use of Theorem 4.7.

Example 4.4 Consider partial metric spaces (X,p), where set $X = \{0,1,2,\ldots\}$ and

$$p(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\}.$$

for all $x,y \in X$. Let $(X,\preceq)$ be a partially ordered set where

$$y \preceq x \implies x \geq y.$$

Define the multivalued function $T : X \to CB^p(X)$ as

$$Tx = \begin{cases} \{x-2,x-1\}, & \text{for } x \geq 2, \\ \\ \{0,x+1\}, & \text{for } x \in \{0,1\}. \end{cases}$$

We note that $x \geq 2$, $x \prec_1 Tx$, $x \preceq y \implies Tx \prec_2 Ty$ and $x \notin Tx$. We define $F \in \mathscr{F}$ as $F(a) = \ln a + a$. The condition (4.27) becomes

$$\frac{H_p(Tx,Ty)}{\mathbb{M}(x,y)} e^{H_p(Tx,Ty) - \mathbb{M}(x,y)} \leq e^{-2\tau}. \tag{4.36}$$

We now calculate $H_p(Tx,Ty)$ for $x > y \geq 2$.

$$Tx = \{x-2, x-1,\}, Ty = \{y-2, y-1,\}.$$

$$p(x-1,y-2) = \frac{3x-y}{4} - \frac{1}{4},\ p(x-1,y-1) = \frac{3x-y}{4} - \frac{1}{2}.$$

$$p(x-2,y-2) = \frac{3x-y}{4} - 1,\ p(x-2,y-1) = \frac{3x-y}{4} - \frac{5}{4}.$$

$$\begin{aligned} p(x-2,Ty) &= \min\{p(x-2,a); a \in Ty\} \\ &= \min\{p(x-2,y-1), p(x-2,y-2)\}, \\ &= \min\{\frac{3x-y}{4} - 1, \frac{3x-y}{4} - \frac{5}{4}\}, \\ &= \frac{3x-y}{4} - \frac{5}{4}. \end{aligned}$$

In a similar manner, we calculate

$$\begin{aligned} p(x-1,Ty) &= \frac{3x-y}{4} - \frac{1}{2}. \\ p(x-2,Tx) &= \frac{3x-y}{4} - 1. \\ p(y-1,Tx) &= \frac{3x-y}{4} - \frac{5}{2}. \end{aligned}$$

$$\begin{aligned} \delta_p(Tx,Ty) &= \max\{p(a,Ty); a \in Tx\} \\ &= \max\{p(x-2,Ty), p(x-1,Ty)\}, \\ &= \max\{\frac{3x-y}{4} - \frac{5}{4}, \frac{3x-y}{4} - \frac{1}{2}\}, \\ &= \frac{3x-y}{4} - \frac{1}{2}. \end{aligned}$$

Similarly

$$\begin{aligned} \delta_p(Ty,Tx) &= \max\{p(a,Tx); a \in Ty\} \\ &= \max\{p(y-2,Tx), p(y-1,Tx)\}, \\ &= \max\{\frac{3x-y}{4} - 1, \frac{3x-y}{4} - \frac{5}{4}\}, \\ &= \frac{3x-y}{4} - 1. \end{aligned}$$

$$\begin{aligned} H_p(Tx,Ty) &= \max\{\delta_p(Tx,Ty),\delta_p(Ty,Tx)\}, \\ &= \max\left\{\frac{3x-y}{4}-\frac{1}{2},\frac{3x-y}{4}-1\right\}, \\ &= \frac{3x-y}{4}-\frac{1}{2}. \end{aligned}$$

We note that

$$\mathbb{M}(x,y)=\alpha p(x,y)=p(x,y)=\frac{3x-y}{4}.$$

Applying to (4.36) we get

$$\frac{3x-y-2}{3x-y}e^{-\frac{1}{2}}\leq e^{-2\tau}.$$

which is true for $\tau=\frac{1}{4}$. The mapping has a fixed point at $x=0$. This shows that T is a multivalued Hardy-Rogers-type F-contraction with contractive factor τ. Hence, satisfy Theorem 4.7.

4.5 Multivalued Jaggi-Hardy-Rogers Type-F-$F_{\mathscr{C}}$ Contraction Results in Partially Ordered b-Metric Spaces

To prove the results of this section we start by combining the concept of definitions (4.1) and (4.11) as follows:

Definition 4.15 *Let (X,d,s) be a partially ordered b-metric space with constant $s\geq 1$. A mapping $T:X\to CB(X)$ is said to be a multivalued generalized Jaggi-Hardy-Rogers type F-contraction if there exists $F\in\mathscr{F}$ and $\tau\in\mathbb{R}^+$ such that*

$$\begin{aligned} &H_b(Tx,Ty)>0 \implies \\ &2\tau+F(sH_b(Tx,Ty)) \leq F\Big(\frac{\eta d_b(x,Tx).d_b(y,Ty)}{d_b(x,y)}+\mu d_b(x,y)+\beta d_b(x,Tx)+ \\ &\qquad \gamma d_b(y,Ty)+\delta d_b(x,Ty)+Ld_b(y,Tx)\Big), \end{aligned} \tag{4.37}$$

for all comparable $x,y\in X$ with $Tx\neq Ty$, where $\eta,\mu,\beta,\gamma,\delta,L\geq 0$, $\eta+\mu+\beta+\gamma+s\delta\leq 1$ and $\gamma\neq 1$.

We prove the following theorem.

Theorem 4.8 Let (X,d_b,s) be a partially ordered b-metric space with coefficient $s\geq 1$. Let $T:X\to CB^b(X)$ be a continuous non dencreasing multivalued mappings satisfying a generalized Jaggi-Hardy-Rogers F-$F_{\mathscr{C}}$- contraction, such that there exists $x_0\in X$ with $x_0\in Tx_0$ and $F\in\mathscr{F}$, $\tau\in\mathbb{R}^+$. Assume that

$$2\tau+F(\phi(sH_b(Tx,Ty))) \leq F(\phi(M(x,y))-\psi(M(x,y))), \tag{4.38}$$

for $\phi\in\Phi,\psi\in\Psi$ and $F_{\mathscr{C}}(\phi,\psi)=\phi-\psi$, where

$$\begin{aligned} M(x,y) &= \frac{\eta d_b(x,Tx).d_b(y,Ty)}{d_b(x,y)}+\mu d_b(x,y)+\beta d_b(x,Tx)+\\ &\quad \gamma d_b(y,Ty)+\delta d_b(x,Ty)+Ld_b(y,Tx), \end{aligned}$$

for all comparable $x,y\in X$ with $Tx\neq Ty$, where $\eta,\mu,\beta,\gamma,\delta,L\geq 0$, $\eta+\mu+\beta+\gamma+s\delta\leq 1$ and $\gamma\neq 1$, $\lambda=\frac{\mu+\beta+s\delta}{s-(\eta+\gamma+s\delta)}<1=\lambda$ and $s\lambda<1$. Then T has a fixed point in X.

Proof. Let x_0 be arbitrary point of X, there exists $x_0\in X$ such that $x_0\prec_1 Tx_0$. Choosing $x_1\in Tx_1$, we have $x_0\preceq x_1\Rightarrow Tx_0\prec_2 Tx_1$. If $x_1\in Tx_1$ then x_1 is a fixed point of T and we have completed the proof.

Suppose $x_1\notin Tx_1$, then $Tx_0\neq Tx_1$. Since F is continuous from the right, there exists a real number $h>1$ (*Remark* 4.2) such that

$$F(hH_b(Tx_0,Tx_1))<F(H_b(Tx_0,Tx_1))+\tau.$$

Now, from $F(d_b(x_1,Tx_1))<F(H_b(Tx_0,Tx_1))$ and $Tx_0\prec_2 Tx_1$, by this case we choose $x_2\in Tx_1$ such that $F(d_b(x_1,x_2))\leq F(H_b(Tx_0,Tx_1))$ and by using Lemma 2.14 as a result

$$d_b(x_1,x_2)\leq hH_b(Tx_0,Tx_1)<H_b(Tx_0,Tx_1)+\tau,$$

implies that

$$F(sd_b(x_1,x_2))\leq F(shH_b(Tx_0,Tx_1))<F(sH_b(Tx_0,Tx_1))+\tau, \tag{4.39}$$

we apply (4.38) and (4.39) with $x=x_0,y=x_1$ to get

$$\begin{aligned} 2\tau+F(\phi(sd_b(x_1,x_2))) &\leq 2\tau+F(\phi(sH_b(Tx_0,Tx_1)))+\tau,\\ &\leq F\big(\phi(M(x_0,x_1))-\psi(M(x_0,x_1))\big)+\tau, \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} M(x_0,x_1) &= \frac{\eta d_b(x_0,Tx_0).d_b(x_1,Tx_1)}{d_b(x_0,x_1)}+\mu d_b(x_0,x_1)+\beta d_b(x_0,Tx_0)+\\ &\quad \gamma p(x_1,Tx_1)+\delta p(x_0,Tx_1)+Ld_b(x_1,Tx_0),\\ &\leq \frac{\eta d_b(x_0,x_1).d_b(x_1,x_2)}{d_b(x_0,x_1)}+\mu d_b(x_0,x_1)+\beta d_b(x_0,x_1)+\\ &\quad \gamma d_b(x_1,x_2)+\delta d_b(x_0,x_2)+Ld_b(x_1,x_1),\\ &\quad \text{by } (B1)\text{, we have,}\\ &\leq \eta d_b(x_1,x_2)+\mu d_b(x_0,x_1)+\beta d_b(x_0,x_1)+\gamma d_b(x_1,x_2)+\delta d_b(x_0,x_2),\\ &\quad \text{by } (B3)\text{, we get,}\\ &\leq \eta d_b(x_1,x_2)+\mu d_b(x_0,x_1)+\beta d_b(x_0,x_1)+\gamma d_b(x_1,x_2)+s\delta d_b(x_0,x_1)+\\ &\quad s\delta d_b(x_1,x_2),\\ &= (\mu+\beta+s\delta)d_b(x_0,x_1)+(\eta+\gamma+s\delta)d_b(x_1,x_2). \end{aligned} \tag{4.41}$$

Using (4.41) in (4.40) we get

$$\begin{aligned}\tau+F(\phi(sd_b(x_1,x_2))) &\leq F\big(\phi((\mu+\beta+s\delta)d_b(x_0,x_1)+(\eta+\gamma+s\delta)d_b(x_1,x_2))-\\ &\qquad \psi((\mu+\beta+s\delta)d_b(x_0,x_1)+(\eta+\gamma+s\delta)d_b(x_1,x_2))\big)\end{aligned} \tag{4.42}$$

As F is an increasing function, by $(F1)$ and property of ϕ and ψ in (4.42) implies

$$\begin{aligned}\Rightarrow sd_b(x_1,x_2) &< (\mu+\beta+s\delta)d_b(x_0,x_1)+(\eta+\gamma+s\delta)d_b(x_1,x_2)\\ \Rightarrow s-(\eta+\gamma+s\delta)d_b(x_1,x_2) &< (\mu+\beta+s\delta)d_b(x_0,x_1),\\ \Rightarrow d_b(x_1,x_2) &< \frac{\mu+\beta+s\delta}{s-(\eta+\gamma+s\delta)}d_b(x_0,x_1).\end{aligned} \tag{4.43}$$

From the assumption we have

$$\lambda = \frac{\mu+\beta+s\delta}{s-(\eta+\gamma+s\delta)} < 1.$$

Hence, (4.43) implies

$$d_b(x_1,x_2) < d_b(x_0,x_1). \tag{4.44}$$

By the property of ϕ, ψ in $F_{\mathscr{C}}$ we have

$$\phi(d_b(x_1,x_2)) \leq \phi(M(x_0,x_1))-\psi(M(x_0,x_1)) < \phi(d_b(x_0,x_1)), \tag{4.45}$$

Using (4.45) in (4.42) we get

$$\begin{aligned}\tau+F(\phi(sd_b(x_1,x_2))) &\leq F(\phi(d_b(x_0,x_1))),\\ F(\phi(sd_b(x_1,x_2))) &\leq F(\phi(d_b(x_0,x_1)))-\tau,\end{aligned} \tag{4.46}$$

Continuing in the same manner we get the sequence $\{x_n\}$ with $x_1 \prec x_2 \prec x_3 \ldots$. If $x_n \in Tx_n$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T and the proof is completed. Suppose $x_n \notin Tx_n$ for all $n \in \mathbb{N}$. In this case we have

$$F(\phi(sd_b(x_n,x_{n+1}))) \leq F(\phi(d_b(x_{n-1},x_n)))-\tau. \tag{4.47}$$

By property $(F4)$ and (4.47) we have

$$F(\phi^n(s^n d_b(x_n,x_{n+1}))) \leq F(\phi^{n-1}s^{n-1}(d_b(x_{n-1},x_n)))-n\tau. \tag{4.48}$$

Let $\alpha = d_b(x_n,x_{n+1}) > 0$ for all $n \in \mathbb{N}$. From (5.35) we have

$$F(\phi^n(s^n\alpha_n)) \leq F(\phi^{n-1}(s^{n-1}\alpha_{n-1}))-n\tau. \tag{4.49}$$

Thus, by (4.49) we get

$$F(\phi^n(s^n\alpha_n)) \leq F(\phi^{n-1}(s^{n-1}\alpha_{n-1}))-\tau \leq \cdots \leq F(\alpha_0))-n\tau. \tag{4.50}$$

We determine the limit $n \to \infty$ of (4.50) and get

$$\lim_{n\to\infty} F\big(\phi^n(s^n\alpha_n)\big) = -\infty.$$

By property $(F2)$, this implies

$$\lim_{n\to\infty} \phi^n(s^n\alpha_n) = 0. \tag{4.51}$$

By condition $(F3)$, there exist $k \in (0,1)$ and such that

$$\lim_{n\to\infty} \left(\phi^n(s^n\alpha_n)\right)^k F\left(\phi^n(s^n\alpha_n)\right) = 0.$$

Multiplying of (4.50) with $\left(\phi^n(s^n\alpha_n)\right)^k$ yields

$$\begin{aligned} \left(\phi^n(s^n\alpha_n)\right)^k F(\phi^n(s^n\alpha_n)) &\leq \left(\phi^n(s^n\alpha_n)\right)^k F(\phi^{n-1}(s^{n-1}\alpha_{n-1})) - \left(\phi^n(s^n\alpha_n)\right)^k \tau \leq \dots \\ &\leq \left(\phi^n(s^n\alpha_n)\right)^k F(\alpha_0)) - \left(\phi^n(s^n\alpha_n)\right)^k n\tau \leq 0. \end{aligned} \tag{4.52}$$

From (4.52), we have

$$\left(\phi^n(s^n\alpha_n)\right)^k F(\phi^n(s^n\alpha_n)) - \left(\phi^n(s^n\alpha_n)\right)^k F(\alpha_0)) \leq -\left(\phi^n(s^n\alpha_n)\right)^k n\tau \leq 0. \tag{4.53}$$

Letting $n \to \infty$ in (4.53), we get

$$\lim_{n\to\infty} n\tau\left(\phi^n(s^n\alpha_n)\right)^k = 0.$$

Consequently,

$$\begin{aligned} \lim_{n\to\infty} (n\tau)^{\frac{1}{k}}\left(\phi^n(s^n\alpha_n)\right) &= 0. \\ \Rightarrow \sum_{n=1}^{+\infty} \phi^n s^n \alpha_n &= 0. \end{aligned}$$

This implies that, there exists $n_1 \in \mathbb{N}$ such that $(n\tau)^{\frac{1}{k}}\left(\phi^n(s^n\alpha_n)\right) < 1$ for all $n > n_1$. Therefore, we have

$$\phi^n(s^n\alpha_n) < \frac{1}{(n\tau)^{1/k}}. \tag{4.54}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Consider $n,m \in \mathbb{N}, n < m < n_1$. By $(B3)$ of Definition 2.22, we have

$$\begin{aligned} d_b(x_n,x_m) &\leq s[d_b(x_n,x_{n+1}) + d_b(x_{n+1},x_m)], \\ &\leq s[d_b(x_n,x_{n+1}) + s[d_b(x_{n+1},x_{n+2}) + d_b(x_{n+2},x_m)]], \\ &\leq s[d_b(x_n,x_{n+1}) + s[d_b(x_{n+1},x_{n+2}) + s[d_b(x_{n+2},x_{n+3}) + d_b(x_{n+3},x_m)]]] \\ &\quad \dots\dots \\ &\leq sd_b(x_n,x_{n+1}) + s^2 d_b(x_{n+1},x_{n+2}) + s^3 d_b(x_{n+2},x_{n+3}) + s^3 d_b(x_{n+3},x_m), \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{s^{n-1}}\Big(\phi^n s^n d_b(x_n,x_{n+1}) + \phi^{n+1}s^{n+1}d_b(x_{n+1},x_{n+2}) + \\
&\quad \phi^{n+2}s^{n+2}d_b(x_{n+2},x_{n+3}) + \phi^{n+2}s^{n+2}d_b(x_{n+3},x_m) + \dots\Big), \\
&= \frac{1}{s^{n-1}}\Big(\phi^n s^n \alpha_n + \phi^{n+1}s^{n+1}\alpha_{n+1} + \phi^{n+2}s^{n+2}\alpha_{n+2} + \\
&\quad \phi^{n+2}s^{n+2}d_b(x_{n+3},x_m) + \dots\Big), \\
&\leq \frac{1}{s^{n-1}}\sum_{i=n}^{m}\phi^i s^i \alpha_i, \\
&\leq \frac{1}{s^{n-1}}\sum_{i=n}^{\infty}\phi^i s^i \alpha_i, \\
&\leq \frac{1}{s^{n-1}}\sum_{i=n}^{\infty}\frac{1}{(i\tau)^{1/k}}, \text{ from (4.54).} \qquad (4.55)
\end{aligned}$$

The series $\sum_{i=m}^{\infty}\frac{1}{i^{1/k}}$ converges as it is a b-series with an exponent greater than one. This implies $\lim_{n,m\to\infty} d_b(x_n,x_m) = 0$. This makes $\{x_n\}$ a Cauchy sequence by (ii) of Definition 2.15. Therefore, we have

$$\lim_{m,n\to\infty} d_b(x_n,x_{n+1}) = 0.$$

If $\{x_{2n}\}$ is not Cauchy sequence in X, then there exists $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive intergers such that the following four sequences

$$d_b(x_{2m_k},x_{2n_k}), d_b(x_{2m_k},x_{2n_{k+1}}), d_b(x_{2m_{k-1}},x_{2n_k}), d_b(x_{2m_{k-1}},x_{2n_{k+1}}),$$

all tend to ε when $k \to \infty$.

Suppose $d_b(x_{2m_k},x_{2n_k}) = \varepsilon$ and $d_b(x_{2m_k},x_{2m_{k+p}}) = 0$ for all $p > 0$, putting $x = x_{2m_k}$ and $y = x_{2n_k}$ in (4.38) we have

$$\tau + F(\phi(sd_b(x_{2m_{k+1}},x_{2n_{k+1}}))) \quad \leq F(\phi(M(x_{2m_k},x_{2n_k})) - \psi(M(x_{2m_k},x_{2n_k}))), \qquad (4.56)$$

where $\phi \in \Phi$, $\psi \in \Psi$ and $F_{\mathscr{C}}(\phi,\psi) = \phi - \psi$ for any $x_0, x_1 \in X$ with $x < x$, we have

$$\begin{aligned}
M(x_{2m_k},x_{2n_k}) &= \frac{\eta d_b(x_{2m_k},Tx_{2m_k}).d_b(x_{2n_k},Tx_{2n_k})}{d_b(x_{2m_k},x_{2n_k})} + \mu d_b(x_{2m_k},x_{2n_k}) + \\
&\quad \beta d_b(x_{2m_k},Tx_{2m_k}) + \gamma d_b(x_{2n_k},Tx_{2n_k}) + \delta d_b(x_{2m_k},Tx_{2n_k}) + \\
&\quad Ld_b(x_{2n_k},Tx_{2m_k}), \\
&= \frac{\eta d_b(x_{2m_k},x_{2m_{k+1}}).d_b(2n_k,x_{2n_{k+1}})}{d_b(x_{2m_k},x_{2n_k})} + \mu d_b(x_{2m_k},x_{2n_k}) + \\
&\quad \beta d_b(x_{2m_k},x_{2m_{k+1}}) + \gamma d_b(x_{2n_k},x_{2n_{k+1}}) + \delta d_b(x_{2m_k},x_{2n_{k+1}}) + \\
&\quad Ld_b(x_{2n_k},x_{2m_{k+1}}),
\end{aligned}$$

$$\begin{aligned} = \quad & \frac{\eta d_b(x_{2m_k}, x_{2m_{k+1}}).d_b(x_{2n_k}, x_{2n_{k+1}})}{d_b(x_{2m_k}, x_{2n_k})} + \mu d_b(x_{2m_k}, x_{2n_k}) + \\ & \beta d_b(x_{2m_k}, x_{2m_{k+1}}) + \gamma d_b(x_{2n_k}, x_{2n_{k+1}}) + \delta d_b(x_{2m_k}, x_{2n_{k+1}}) + \\ & L d_b(x_{2n_k}, x_{2m_{k+1}}). \end{aligned}$$

Taking $k \to \infty$ in the above inequality and $\mu + \delta + L < 1$ yields to

$$\begin{aligned} M(x_{2m_k}, x_{2n_k}) &= \mu d_b(x_{2m_k}, x_{2n_k}) + \delta d_b(x_{2m_k}, x_{2n_{k+1}}) + L d_b(x_{2n_k}, x_{2m_{k+1}}), \\ &\leq \mu(\varepsilon) + \delta(\varepsilon) + L(\varepsilon) = (\mu + \delta + L)\varepsilon, \\ &= \varepsilon. \end{aligned} \tag{4.57}$$

Using (4.57) in (4.56), we get

$$\tau + F(\phi(s\varepsilon)) \leq F(\phi(\varepsilon) - \psi(\varepsilon)).$$

From (5.33) and $(F1)$, leads to

$$F(\phi(s\varepsilon)) \leq F(\phi(\varepsilon)) - \tau. \tag{4.58}$$

We notice that (4.58) is identical to (4.46). Next, proceeding as in the proof of Theorem 4.8, we obtain that $\{x_{2n}\}$ converge, which is a contradiction. Hence, $\{x_{2n}\}$ is a cauchy sequence.

Similarly, by Lemma 2.3 and 2.4 one can show the given four sequences are Cauchy sequences in b-metric space as below:

Assume that

$$n_k > m_k, d_b(x_{2m_k}, x_{2n_{k-1}}) < \varepsilon, d_b(x_{2m_k}, x_{2n_k}) > \varepsilon,$$
$$\lim_{k\to\infty} d_b(x_{2m_k}, x_{2m_k}) = 0.$$

By $(B3)$, we have

$$\begin{aligned} \varepsilon \leq d_b(x_{2m_k}, x_{2n_k}) &\leq s[d_b(x_{2m_k}, x_{2n_{k-1}}) + d_b(x_{2n_{k-1}}, x_{2n_k})], \\ &\leq s d_b(x_{2m_k}, x_{2n_{k-1}}) + s d_b(x_{2n_{k-1}}, x_{2n_k}), \\ &\leq s\varepsilon + s d_b(x_{2n_{k-1}}, x_{2n_k}). \end{aligned}$$

Taking the upper limit and lower limit as $k \to \infty$ in (5.55), and using (5.54) we obtain

$$\varepsilon \leq d_b(x_{2m_k}, x_{2n_k}) \leq \lim_{n\to+\infty} \inf d_b(x_{2m_k}, x_{2n_k}) \leq \lim_{n\to+\infty} \sup d_b(x_{2m_k}, x_{2n_k}) \leq s\varepsilon. \tag{4.59}$$

Using $(B3)$ again and (5.56), we have

$$\begin{aligned} \varepsilon \leq d_b(x_{2m_k}, x_{2n_k}) &\leq s[d_b(x_{2m_k}, x_{2n_{k+1}}) + d_b(x_{2n_{k+1}}, x_{2n_k})], \\ &\leq s[s d_b(x_{2m_k}, x_{2n_k}) + s d_b(x_{2n_k}, x_{2n_{k+1}})] + s d_b(x_{2n_{k+1}}, x_{2n_k}), \\ &\leq s^2 d_b(x_{2m_k}, x_{2n_k}) + s^2 d_b(x_{2n_k}, x_{2n_{k+1}}) + s d_b(x_{2n_{k+1}}, x_{2n_k}), \\ &\leq s^3\varepsilon. \end{aligned}$$

Equivalently,

$$\frac{\varepsilon}{s} \leq d_b(x_{2m_k}, x_{2n_{k+1}}) \leq \lim_{n\to+\infty} \inf d_b(x_{2m_k}, x_{2n_{k+1}}) \leq \lim_{n\to+\infty} \sup d_b(x_{2m_k}, x_{2n_{k+1}}) \leq s^2\varepsilon.$$

The remaining two sequences can be proved in a similar way giving:

$$\begin{aligned}
\frac{\varepsilon}{s} \leq d_b(x_{2m_{k-1}}, x_{2n_k}) &\leq \lim_{n\to+\infty} \inf d_b(x_{2m_{k-1}}, x_{2n_k}) \\
&\leq \lim_{n\to+\infty} \sup d_b(x_{2m_{k-1}}, x_{2n_k}) \leq s^2\varepsilon. \\
\frac{\varepsilon}{s^2} \leq d_b(x_{2m_{k-1}}, x_{2n_{k+1}}) &\leq \lim_{n\to+\infty} \inf d_b(x_{2m_{k-1}}, x_{2n_{k+1}}) \\
&\leq \lim_{n\to+\infty} \sup d_b(x_{2m_{k-1}}, x_{2n_{k+1}}) \leq s^3\varepsilon
\end{aligned}$$

which is a contradiction. Hence, $\{x_{2n}\}$ is a Cauchy sequence. In this way we proved that $\{x_n\}$ is a Cauchy sequence. Also, since (X, d_b) is a complete partially ordered b-metric space, there is $x^\star \in X$ such that $x_n \to x^\star$, and $d_b(x^\star, x^\star) = 0$. We claim that $x^\star$ is a fixed point of T. We do this by showing that $d_b(x^\star, Tx^\star) = d_b(x^\star, x^\star) = 0$. Suppose $d_b(x^\star, Tx^\star) > 0$. Then there exists some $n_0 \in \mathbb{N}$ such that $d_b(x_n, Tx^\star) > 0$ for all $n > n_0$.

By (6.43) with $x = x_n$ and $y = x^*$, we get

$$2\tau + F(\phi(sd_b(x_{n+1}, Tx^*))) \leq F\big(\phi(M(x_n, x^*)) - \psi(M(x_n, x^*))\big) + \tau, \quad (4.60)$$

where

$$\begin{aligned}
M(x_n, x^*) &= \frac{\eta d_b(x_n, Tx_n).d_b(x^*, Tx^*)}{d_b(x_n, x^*)} + \mu d_b(x_n, x^*) + \beta d_b(x_n, Tx_n) + \\
&\quad \gamma d_b(x^*, Tx^*) + \delta d_b(x_n, Tx^*) + Ld_b(x^*, Tx_n), \\
&\leq \frac{\eta d_b(x_n, x_{n+1}).d_b(x^*, Tx^*)}{d_b(x_n, x^*)} + \mu d_b(x_n, x^*) + \beta d_b(x_n, x_{n+1}) + \\
&\quad \gamma d_b(x^*, Tx^*) + \delta d_b(x_n, Tx^*) + Ld_b(x^*, x_{n+1}). \qquad (4.61)
\end{aligned}$$

Taking limit as $n \to \infty$ in (5.58) we obtain

$$\begin{aligned}
M(x_n, x^*) &= \frac{\eta d_b(x^*, x^*).d_b(x^*, Tx^*)}{d_b(x^*, x^*)} + \mu d_b(x^*, x^*) + \beta d_b(x^*, x^*) + \\
&\quad \gamma d_b(x^*, Tx^*) + \delta d_b(x^*, Tx^*) + Ld_b(x^*, x^*), \\
&\Longrightarrow (\eta + \gamma + \delta) d_b(x^*, Tx^*). \qquad (4.62)
\end{aligned}$$

Using (4.62) in (4.60), we get

$$\begin{aligned}
\tau + F(\phi(sd_b(x_{n+1}, Tx^*))) &\leq F\big(\phi((\eta + \gamma + \delta) d_b(x^*, Tx^*)) - \\
&\quad \psi(\eta + \gamma + \delta) d_b(x^*, Tx^*))\big).
\end{aligned} \qquad (4.63)$$

Again, taking limit as $n \to \infty$ in (4.63) and applying the fact that F is an increasing function, we get

$$\begin{aligned}\tau + F(\phi(sd_b(x^*,Tx^*))) &\leq F\big(\phi((\eta+\gamma+\delta)d_b(x^*,Tx^*)) - \psi(\eta+\gamma+\delta)d_b(x^*,Tx^*))\big),\\ &\leq F(\phi(d_b(x^*,Tx^*))) < d_b(x^*,Tx^*).\end{aligned}$$

By the properties of F and $F_{\mathscr{C}}$, since they are increasing functions, unless $d_b(x^*,Tx^*) = 0$.

As a result $\tau \leq 0$, which is a contradiction. Hence $d_b(x^\star,Tx^\star) = 0$ making $x^\star$ a fixed point of T.

Finally, we claim that the fixed point obtained is unique. Let y^* be another fixed point of T such that $\lim_{n\to\infty} x_n \to y^*$ and assume that $d_b(x^*,y^*) > 0$. To see this, by (B3) with $x = x^*, y = y^*$, we have

$$\begin{aligned}d_b(x^*,y^*) &\leq s[d_b(x^*,x_{n+1}) + d_b(x_{n+1},y^*)],\\ &\leq sd_b(x^*,y^*) + ds_b(y^*,y^*),\\ &\leq sd_b(x^*,y^*),\\ d_b(x^*,y^*) - sd_b(x^*,y^*) &\leq 0,\\ (1-s)d_b(x^*,y^*) &\leq 0,\\ d_b(x^*,y^*) &\leq 0, \qquad (4.64)\end{aligned}$$

Which is a contradiction. Hence, $x^* = y^*$; that is x^* is a unique fixed point of T. □

We give the following corollary, which is inspired by Kumar (2019).

Corollary 4.1 Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a b-metric space on X such that (X,d_b,s) is a complete b-metric space. Suppose T and g are continuous multivalued F-$F_{\mathscr{C}}$-contraction mappings on X, $T(X) \subseteq g(X)$, T is monotone g-non decreasing mapping and a generalized Jaggi-Hardy-Rogers ϕ-ψ-$F_{\mathscr{C}}$-F-contraction

$$2\tau + F(\phi(sd_b(Tx,Ty))) \leq F(\phi(M(x,y)) - \psi(M(x,y))), \qquad (4.65)$$

where $\phi \in \Phi$, $\psi \in \Psi$ and $F_C(\phi,\psi) = \phi - \psi$ for any $x,y \in X$ with $x < y$,

$$\begin{aligned}M(x,y) &= \frac{\eta d_b(gx,Tx).d_b(gy,Ty)}{d_b(gx,gy)} + \mu d_b(gx,gy) + \beta d_b(gx,Tx) +\\ &\quad \gamma d_b(gy,Ty) + \delta d_b(gx,Ty) + Ld_b(gy,Tx),\end{aligned}$$

for all comparable $x,y \in X$ with $Tx \neq Ty$, where $\eta,\mu,\beta,\gamma,\delta,L \geq 0$, $\eta+\mu+\beta+\gamma+s\delta \leq 1$ and $\gamma \neq 1$, for which gx and gy are comparable and $\tau > 0$. If there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$ and T and g are compatible, then T and g have a coincident point.

Proof. Let x_0 be an arbitrary point in X. Suppose $gx_0 \preceq Tx_0$ and $TX \subseteq g(X)$, we construct a Picard sequence by choosing $x_1 \in X$ so that $gx_1 = Tx_1$. By Induction we can construct a sequence $\{x_n\}$ such that $gx_{n+1} = Tx_n$ for every $n \geq 0$. Using $x = x_n$ and $y = x_{n+1}$ in (4.65). The other steps follow similar proof of Theorem 4.8. This complete the proof. □

By extending Theorem 4.3, we prove following corollary:

Corollary 4.2 Let (X, d_b, s) be a complete b-metric space with a constant $s \geq 1$ and $T : X \to CB^b(X)$ be an σ-F-$F_{\mathscr{C}}$-multivalued mappings. Assume that there exists $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\sigma(x,y) + F(sH_b(Tx,Ty)) \leq F(\psi(M(x,y)), \phi(M(x,y))), \tag{4.66}$$

where

$$M(x,y) = \left\{ d_b(x,y), \frac{d_b(x,fx)}{1+d_b(x,fx)}, \frac{d_b(y,fy)}{1+d_b(y,fy)}, \frac{1}{2}[d_b(x,fy)+d_b(y,fx)] \right\},$$

for all $x,y \in X$ assume further that

- **(i)** T is σ-contraction of type s and ψ is continuous;
- **(ii)** there exists $x_0 \in X$ and $x_1 \in Tx_0$ with $\sigma(x_0,x_1) \geq s$;
- **(iii)** $\{x_n\}$ a sequence in X with $x_n \to x$ as $n \to \infty$ and $\sigma(x_n,x_{n+1}) \geq s$ forall $n \in \mathbb{N} \cup \{0\}$, then $\sigma(x_n,x) \geq s$ forall $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Proof. Using condition (ii), let $x_0 \in X$ and $x_1 \in Tx_0$ such that $\sigma(x_0,x_1) \geq s$, if $x_0 = x_1$ or $x - 1 \in Tx_1$, we can conclude that x_1 is a fixed point of T. Hence the proof is completed. Suppose on contrary that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. By applying $x = x_0$ and $y = x_1$ in (4.66) with $\sigma(x_0,x_1) > s$ and following the similar procedure for the proof of Theorem 4.8, this complete the proof. □

Next, we give an illustrative example to validate the conditions imposed in Theorem 4.8.

Example 4.5 Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ be endowed with partially ordered b-metric $d_b(x,y) = (x-y)^2$ with $s = 2$ for all $x,y \in X$ with $x = \frac{1}{n}$ and $y = \frac{1}{m}$. Define $T : X \to CB(X)$ by

$$Tx = \left[0, \frac{1}{n^3[x - (\frac{1}{n})]^2 + 1}\right]$$

, $\forall x,y \in X$. Furthermore, define the functions $\phi, \psi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t$ and $\psi(t) = \frac{1}{4}t$. Take $F(a) = \ln a$, $\tau > 0$ for all $a \in X$.

We prove that the conditions imposed on T in Theorem 4.8 are satisfied:

The condition of (F) translate (4.37) to

$$\begin{aligned} \phi(sH_b(Tx,Ty)) &\leq e^{-2\tau}\phi(M(x,y))-\psi(M(x,y)). \\ \Longrightarrow \frac{\phi(sH_b(Tx,Ty))}{\phi(M(x,y))-\psi(M(x,y))} &\leq e^{-2\tau}. \end{aligned} \tag{4.67}$$

Now, our task is to verify inequality (4.67). To see this, we have one case to investigate: that is, for $x=\frac{1}{n}$ and $y=\frac{1}{m}$ such that $m>n$, $\forall\ m,n\in\mathbb{N}$ in (4.67). Let $n=1,m=2$ and $s=2$. If $Tx=Ty$, so $H(Tx,Ty)=0$, that is, (5.49) is satisfied. Otherwise, we commerce by calculating the following metrics

$$\begin{aligned} H_b(Tx,Ty) &= H_b\Bigg(\Big[0,\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\Big],\Big[0,\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\Big]\Bigg) \\ &= d_b\Bigg(\Big[0,\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\Big],\Big[0,\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\Big]\Bigg) \\ &= \Big|\frac{1}{n^3[x-(\frac{1}{n})]^2+1}-\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\Big|^2 \\ &= \Big|\frac{[y-(\frac{1}{n})]^2-[x-(\frac{1}{n})]^2}{n^3([x-(\frac{1}{n})]^2+1)([y-(\frac{1}{n})]^2+1)}\Big|^2 \\ &= \Big|\frac{[\frac{1}{m}-(\frac{1}{n})]^2-[\frac{1}{n}-(\frac{1}{n})]^2}{n^3([\frac{1}{n}-(\frac{1}{n})]^2+1)([\frac{1}{m}-(\frac{1}{n})]^2+1)}\Big|^2 \\ &= \frac{1}{25}. \\ d_b(x,y) &= d_b\Big(\frac{1}{n},\frac{1}{m}\Big)=\Big|\frac{1}{n}-\frac{1}{m}\Big|^2=\frac{1}{4}, \\ d_b(x,Tx) &= d_b\Bigg(x,\Big[0,\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\Big]\Bigg) \\ &= \Big|x-\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\Big|^2 \\ &= \Big|\frac{1}{n}-\frac{1}{n^3[\frac{1}{n}-(\frac{1}{n})]^2+1}\Big|^2 \\ &= \Big|\frac{1}{1}-\frac{1}{1^3[\frac{1}{1}-(\frac{1}{1})]^2+1}\Big|^2 \\ &= 0. \end{aligned}$$

$$
\begin{aligned}
d_b(y,Ty) &= d_b\left(y,\left[0,\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\right]\right)\\
&= \left|y-\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\right|^2\\
&= \left|\frac{1}{m}-\frac{1}{n^3[\frac{1}{m}-(\frac{1}{n})]^2+1}\right|^2\\
&= \left|\frac{1}{2}-\frac{1}{1^3[\frac{1}{2}-(\frac{1}{1})]^2+1}\right|^2\\
&= \frac{9}{100}.\\
d_b(x,Ty) &\leq d_b\left(x,\left[0,\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left|x-\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\right|^2\\
&= \left|\frac{1}{n}-\frac{1}{n^3[\frac{1}{m}-(\frac{1}{n})]^2+1}\right|^2\\
&= \left|\frac{1}{1}-\frac{1}{1^3[\frac{1}{2}-(\frac{1}{1})]^2+1}\right|^2\\
&= \frac{1}{25}.\\
d_b(y,Tx) &= d_b\left(y,\left[0,\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\right]\right)\\
&= \left|y-\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\frac{1}{m}-\frac{1}{n^3[\frac{1}{n}-(\frac{1}{n})]^2+1}\right|^2\\
&= \left|\frac{1}{2}-\frac{1}{1^3[\frac{1}{1}-(\frac{1}{1})]^2+1}\right|^2\\
&= \frac{1}{4}.
\end{aligned}
$$

Suppose $\eta=\mu=\frac{1}{7},\beta=\frac{1}{10},\gamma=\frac{1}{6},\delta=\frac{1}{4},L=0$ and using the above inequalities we obtain

$$\begin{aligned} M(x,y) &= \eta\frac{\Big[x-\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\Big]^2\times\Big[y-\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\Big]^2}{(x-y)^2}+\mu(x-y)^2+\\ &\quad \beta\Big[x-\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\Big]^2+\gamma\Big[y-\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\Big]^2+\\ &\quad \delta\Big[x-\frac{1}{n^3[y-(\frac{1}{n})]^2+1}\Big]^2+L\Big[y-\frac{1}{n^3[x-(\frac{1}{n})]^2+1}\Big]^2, \end{aligned}$$

$$\begin{aligned} \Longrightarrow M(x,y) &= \eta\frac{0\times\frac{9}{100}}{\frac{1}{4}}+\mu\frac{1}{4}+\beta\times 0+\gamma\times\frac{9}{100}+\delta\times\frac{1}{25}+L\times\frac{1}{4},\\ &= \frac{1}{6}\Big(\frac{0\times\frac{9}{100}}{\frac{1}{4}}\Big)+\frac{1}{7}\times\frac{1}{4}+\frac{1}{10}\times 0+\frac{1}{6}\times\frac{9}{100}+\frac{1}{4}\times\frac{1}{25}+0\times\frac{1}{4},\\ &= 0.150714285 \end{aligned}$$

By using the above inequalities in (4.67) yields

$$\begin{aligned} \frac{\phi(2\times 0.04)}{\phi(0.150714285)-\psi(0.150714285)} &\leq e^{-2\tau},\\ \Longrightarrow \frac{\phi(2\times 0.04)}{\frac{3}{4}\phi(0.150714285)} &\leq e^{-2\tau}, \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{0.1066666}{0.150714285} &\leq e^{-2\tau},\\ \Longrightarrow 0.707740919 &\leq e^{-2\tau},\\ \Longrightarrow \ln(0.707740919) &\leq \ln(e^{-2\tau}),\\ \Longrightarrow -0.345677185 &\leq -2\tau,\\ 0.172838592 &\leq \tau, \end{aligned}$$

is true, for $\tau > 0$. Hence T satisfies contraction condition (4.38).

4.6 Some Applications

A differential equation, if it has a solution then one must check whether the solution is unique. This section has three subsections. First, in subsection 4.6.1, we give an application for Existence solution for Volterra type integral equation using Hardy-Rogers contraction mapping in ordered partial metric space for demonstration of Theorem 4.9. Then, in subsection 4.6.2, we give an application to the first-order nonlinear differential equation using Theorem 4.9. Next, in Subsection 4.6.3, we investigate some unique solutions to the initial valued problem as the application of Theorem 4.8 in the setting of b-metric space.

4.6.1 Existence Solution for Volterra Type Integral Equation

In this section, we will provide an application of Theorem 4.7 for Hardy Rogers type contraction in ordered partial metric spaces. We will use Volterra type integral equation to illustrate the results. Consider the Volterra type integral equation :

$$x(t) = g(t) + \int_0^t f(t,s,x(s))ds, \; t \in [0,K], \tag{4.68}$$

where $K > 0$. Let $X = C([0,K],\mathbb{R})$ be the space of all continuous functions defined on $[0,K]$. Notice that $(C([0,K])$ endowed with partial metric.

$$p(x,y) = \|x-y\|_\infty = \max_{t\in[0,K]} |x(t) - y(t)|, \tag{4.69}$$

is a complete partial metric space and X can be equipped with the partial order $\preccurlyeq$ given by $x,y \in X$, $(x \preceq y) \implies (x(t) \preccurlyeq y(t)$ and $\|x\|_\infty, \| y\|_\infty \leq 1)$, or $(x(t) = y(t))$ for all $t \in [0,K]$. It was shown by Nieto and Rodrigurz-Lopez (2005) that $(X,\preccurlyeq)$ is regular. From Equation 4.68, x is the solution of $x^{'}(t) = f(t,y(s))$ satisfying initial condition $x(t_0) = x_0$ if and only if

$$x(t) = g(t) + \int_0^t f(t,s,x(s))ds, \; t \in [0,K]. \tag{4.70}$$

We consider Volterra integral equation as

$$\begin{cases} x^{'}(t) = f(t,x(s)), \\ x(t_0) = x_0. \end{cases}$$

Equation (4.70) may be formulated as a fixed point equation

$$x = Tx.$$

Let $\ll$ be a partial order relation on $\mathbb{R}$. Define a mapping $T : X \to X$ by

$$Tx(t) = g(t) + \int_0^t f(t,s,x(s))ds, \; t \in [0,K]. \tag{4.71}$$

Theorem 4.9 *Let* $X= C([0,K]\times\mathbb{R},\mathbb{R})$ *for all value* $x,y\in X$ *;*

(i) $f(t,s,x(s)):\mathbb{R}\to\mathbb{R}$ *is increasing for all* $t\in[0,K]$ *and for* $x,y\in X$, $x\prec y\Leftrightarrow Tx\prec_1 Ty$*;*

(ii) *there exists* $x_0\in X$ *such that* $x_0\prec_1 Tx_0$*;*

$$x_0(t)\prec_1\ g(t)+\int_0^t f(t,s,x(s))ds,\ t\in[0,K]:$$

(iii) *there exist* $\tau\in[1,\infty]$ *such that*

$$|f(t,s,x(s))-f(t,s,y(s))|\leq L(t,s)|x(s)-y(s)|,$$

where $L(t,s)=\alpha\tau e^{-2\tau}$*, for all* $t\in[0,K]$ *and* $x,y\in\mathbb{R}$ *with* $x\prec y$.

(iv) *if* $x_n\to x$ *is a non decreasing sequence in* X*, for all* n *and*

$$2\tau+F(H_p(Tx,Ty))\leq F(\mathbb{M}(x,y)),\tag{4.72}$$

where

$$\mathbb{M}(x,y)=\alpha\ p(x,y)+\beta p(x,Tx)+\gamma p(y,Ty)+\delta p(x,Ty)+\ Lp(y,Tx)$$

for $x,y\in X$, $\tau>0$, $\alpha,\beta,\delta\leq$, $L\geq 0$, $\alpha+\beta+\gamma+2\delta=1$ *and* $\gamma\neq 1$. *Then* T *has a fixed point. Therefore Equation* (4.68) *has at least one fixed point* $x\in X$.

Proof. Using (i), let K be a kernel function on $G=[0,K]\times[0,K]$ and is increasing on G. Then is bounded function on G. For $t,s\in[0,K]$, where $K:[0,K]\times[0,K]\times\mathbb{R}\to\mathbb{R}$ and $f(t),x(s),y(s):[0,K]\to\mathbb{R}$ are continuous functions. Hence $x\prec y\Leftrightarrow Tx\prec_2 Ty$. From (ii) take $x_0\in X$ as an initial point on $[0,K]$ we note that there is point x^* which is the limit of iterative sequence $(x_0,x_1,x_2,x_3,...x_{n+1})$ where x_0 is any continuous function on X and for $(n=0,1,2,...)$, we have

$$x_{n+1}(t)=\ g(t)+\int_0^t f(t,s,x(s))ds,\ t\in[0,K].$$

Suppose we start with $x_0=1=g(t)$ we get the following iteration of a sequence

$$\begin{aligned}
x_1(t) &= 1+\int_{t_0}^t 1.ds=1+t,\\
x_2(t) &= 1+\int_0^t x_1(s)ds=1+t+\frac{t^2}{2},\\
x_3(t) &= 1+\int_0^t x_2(s)ds=1+t+\frac{t^2}{2}+\frac{t^3}{6},\\
\dots &= \dots\\
x_n(t) &= 1+\int_0^t x_{n-1}(s)ds=\sum_{n=0}^{n}\frac{t^n}{n!}.
\end{aligned}$$

The limit of this sequence

$$\lim_{n\to\infty} x_n(t) = e^t,\ \forall\, t \in [0,K].$$

For arbitrary $x \in X$, define $|x|_\tau = \max\limits_{t\in[0,K]}\{|x|e^{-\tau t}\}$, where $\tau \geq 1$ is taken randomly. Since $\|.\|_\tau$ is a Banach space norm equivalent to maximum norm and $(X, \|.\|_\tau)$ endowed with a metric d_τ given as below by O'Regan and Petrusel (2008). Also one can see (Sgroi and Vetro 2013).

$$d_\tau(x,y) \ = \ \max_{t\in[0,K]}\{|x(t) - \ y(t)|\}e^{-\tau t}, \tag{4.73}$$

for all $x, y \in X$. Next, assume that X endowed with partial metric defined by Paesano and Vetro (2014) as follows:

$$p_\tau(x,y) = \begin{cases} d_\tau(x,y) & if \ \ \|x\|_\infty, \|y\|_\infty \leq 1, \\ d_\tau(x,y) + \tau & otherwise. \end{cases}$$

Therefore (X, p) is 0 - complete partial metric. Also

$$p_\tau(x,y) = \begin{cases} d_\tau(x,y) & if \ \|x\|_\infty, \|y\|_\infty \leq 1 \ or \ (\|y\|_\infty > 1), \\ d_\tau(x,y) + \tau & otherwise, \end{cases}$$

and consequently (X, p^s_τ) is 0 - complete. Consider partial order defined on X by $x, y \in C([0,K] \times \mathbb{R}^n, \mathbb{R})$, $x \preceq y$ if and only if $x(t) \preceq \ y(t)$, for $t \in [0,K]$. Then $(X, \|.\|_\tau), \preceq)$ is complete partial ordered metric space and for any increasing sequence $\{x_n\}$ in X, it has the limit $x^* \in X$, we have $x_n \preceq x^*$ for any $t \in [0,K]$.

Assume that the initial condition of Equation 4.70 is $x_0(t) = x_0$ for $t \in [0,K]$ has a unique solution. The solution of Volterra equation is the fixed point of T. Thus, (i) and (ii) are satisfied. From condition (iv) the operator T is surely increasing. Now we have to justify condition of Equation (4.72) by comparing $Tx \prec_2 Ty$ and $x, y \in X$ such that

$x \preceq y$. On using condition (i) and (iii), we reach on the following results

$$\begin{aligned}
|Tx(t)-Ty(t)| &= |\int_0^t f(t,s,x(s)ds) - \int_0^t f(t,s,y(s)ds| \\
&\leq \int_0^t |f(t,s,x(s)) - f(t,s,y(s))|ds \\
&\leq \alpha\tau e^{-2\tau}\int_0^t |x(s)-y(s)|ds \\
&\leq \alpha\tau e^{-2\tau}\int_0^t |x(s)-y(s)|e^{-\tau s}e^{\tau s}ds \\
&\leq \alpha\tau e^{-2\tau}\int_0^t e^{\tau s}|x(s)-y(s)|e^{-\tau s}ds \\
&\leq \alpha\tau e^{-2\tau}\left(\int_0^t e^{\tau s}ds\right)|x(s)-y(s)|e^{-\tau s} \\
&\leq \alpha\tau e^{-2\tau}\left(\int_0^t e^{\tau s}ds\right)\|x(s)-y(s)\|_\tau \\
&\leq \alpha\tau e^{-2\tau}\frac{e^{\tau t}}{\tau}\|x(s)-y(s)\|_\tau, \\
&\leq \alpha e^{-2\tau}\|x(s)-y(s)\|_\tau.
\end{aligned}$$

After all, since $x, y \in X$ such that $x \preccurlyeq y$, from $\|x\|_\tau, \|y\|_\tau \leq 1$, we have

$$|Tx(t)-Ty(t)|e^{-\tau t} \leq \alpha e^{-2\tau}\|x-y\|_\tau,$$

or equivalently,

$$p_\tau(Tx,Ty) \leq \alpha e^{-2\tau} p_\tau(x,y).$$

Taking naturai logarithm to both sides, we obtain

$$\ln(p_\tau(Tx,Ty)) \leq \ln(\alpha e^{-2\tau} p_\tau(x,y)),$$

which is equivalently,

$$2\tau + F(p_\tau(Tx,Ty)) \leq F(\alpha p_\tau(x,y)).$$

for $\alpha = 1$, we have

$$2\tau + F(p_\tau(Tx,Ty)) \leq F(p_\tau(x,y)).$$

Through observation for a function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(a) = \ln a$, for all $x \in X$, belong to $\mathscr{F}$ and so we deduce that the operator T satisfies condition of Equation (4.70) with $\alpha = 1$ and $\beta = \gamma = \delta = 0, L = 0$. Hence by Theorem 4.7, we obtain that operator T has a fixed point $x^* \in X$, which is the solution of Volterra integral Equation (4.68). $\square$

4.6.2 Existence Theorem for the Solution of First-Order Ordinary Differential Equations in Partially Ordered b-Metric spaces

This section investigates the existence of the solution to the first-order periodic problem in the setting of b-metric spaces. Consider the space $C(I)$, the class of real-valued continuous functions defined on $I = [0,T]$ endowed with the metric d_b given by

$$d_b(u,v) = \sup_{t\in I}\Big\{|u(t)-v(t)|^2 : t \in I\Big\},$$

for all $x,y \in C(I)$.

Note that $(C(I),d_b,s)$ is a complete partially ordered b-metric space. Furthermore, the partial order $C(I)$ is defined by

$$x,y \in C(I), u \leq v \implies u(t) \leq v(t),$$

for $t \in I$.

Consider the first order periodic differential problem

$$\begin{cases} u^{'}(t) = f(t,u(t)),\ t \in (0,T), \\ u(0) = u(T), \end{cases} \tag{4.74}$$

where $T > 0,\ f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a continuous function. We consider a space $C(I)(I = [0,T])$ of continuous function defined on I.

Yan *et al.* (2012) proved the existence of the lower solution to ordinary differential equations in partial ordered metric spaces as follows:

Definition 4.16 (Yan *et al.* 2012) A lower solution for (4.74) is a function $\alpha \in C^1(I)$ such that

$$\begin{cases} \alpha^{'}(t) \leq f(t,\alpha(t)) \quad \text{for } t \in I, \\ \\ \alpha(0) \leq \alpha(T). \end{cases}$$

Theorem 4.10 (Yan *et al.* 2012) Consider the problem (4.74) with $f : I\times\mathbb{R} \to \mathbb{R}$ continuous, and suppose that there exists $\lambda,\alpha > 0$ with

$$\alpha \ \leq \ \left(\frac{2\lambda(e^{\lambda T}-1)}{T(e^{\lambda T}+1)}\right)^{\frac{1}{2}},$$

such that for $u, v \in \mathbb{R}$ with $u \geq v$.

$$0 \leq f(t,u) + \lambda u - [f(t,v) + \lambda v] \leq \alpha \sqrt{\ln[(u-v)^2 + 1]},$$

Then the existence of lower solution for (4.74) provide the existence of a unique solution.

As motivated by Yan *et al.* (2012) we will prove the solution of the first-order periodic problem in the b-metric spaces context. Now we extend Theorem 4.10 to b-metric spaces.

Theorem 4.11 Consider the problem (4.74) with $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous, and assume that there exist $\tau > 0, \lambda, \eta > 0$ with

$$\eta \leq \sqrt{\left(\frac{2\lambda(e^{\lambda T} - 1)}{T(e^{\lambda T} + 1)}\right)}, \tag{4.75}$$

such that for $x, y \in \mathbb{R}$ with $x > y$.

$$0 \leq |f(t,u) + \lambda u - [f(t,v) + \lambda v]| \leq e^{-2\tau} \eta \sqrt{\ln[(u-v)^2 + 1]}, \tag{4.76}$$

where

$$\sqrt{\ln[(u-v)^2 + 1]} = M(u,v) = \frac{\eta d_b(u,Tv).d_b(v,Tv)}{d_b(u,v)} + \mu d_b(u,v) + \beta d_b(u,Tu) + \gamma d_b(v,Tv) + \delta d_b(u,Tv) + L d_b(v,Tu),$$

Then the existence of the lower solution for (4.74) provides the existence of a unique solution.

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)[f(s,u(s) + \lambda u(s))]ds, \tag{4.77}$$

where the Green function associated with the above integral equation is given by

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \leq s \leq t \leq T, \\ \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \leq t \leq s \leq T. \end{cases}$$

Proof: Consider $C^1([a,b],X)$ with the metric

$$d_b(u,v) = \sup_{t \in I} |u(t) - v(t)|^2.$$

Then (X, d_s) is a complete b-metric space.

Let $T : C^1([a,b]) \to C^1([a,b])$ be an operator defined by

$$Tu(t) = \int_0^T G(t,s)[f(s,u(s)+\lambda u(s))]ds. \tag{4.78}$$

It is true that $u \in C(I)$ is a solution of (4.74) if $u \in C(I)$ is a fixed point of T. Our objective is to check for conditions imposed in Theorem 4.8 are satisfied with a lower solution in (4.74). Since our mapping T is continuous, let $u > v$ we obtain

$$f(t,u) + \lambda u \geq f(t,v) + \lambda v$$

Using (4.76), we have

$$Tu(t) = \int_0^T G(t,s)[f(s,u(s)+\lambda u(s))]ds \geq \int_0^T G(t,s)[f(s,v(s)+\lambda v(s))]ds = Tv(t),$$

for $t \in I$. Consider $u > v$ and (4.78) we get

$$\begin{aligned} H_b(Tu,Tv) &= \left(\sup_{t\in I} |Tu(t) - Tv(t)|\right)^2, \\ &\leq \left(\sup_{t\in I} \int_0^T G(t,s)[f(s,u(s)+\lambda u(s)) - f(s,v(s)-\lambda v(s))]ds\right)^2, \\ &\leq \left(\sup_{t\in I} \int_0^T G(t,s)e^{-\tau}\eta\sqrt{\ln[(u-v)^2+1]}ds\right)^2. \end{aligned} \tag{4.79}$$

Recall Cauchy-Schwartz inequality defined by

$$\left(\sum_{i=1}^{\infty} |x_i y_i|\right)^2 \leq \left(\sum_{i=1}^{\infty} |x_i|^2\right)\left(\sum_{i=1}^{\infty} |y_i|^2\right) \tag{4.80}$$

Using (4.80) in (4.79) we obtain

$$\begin{aligned} \int_0^T G(t,s)\eta\sqrt{\ln[(u(s)-v(s))^2+1]} &\leq \left(\int_0^T G(t,s)^2\right)^{\frac{1}{2}} \\ &\qquad \left(\int_0^T e^{-2\tau}\eta\sqrt{\ln[(u(s)-v(s))^2+1]}\right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{aligned}
\int_0^T G(t,s)^2 &= \int_0^t G(t,s)^2 ds + \int_t^T G(t,s)^2 ds, \\
&= \int_0^t \frac{e^{2\lambda(T+s-t)}}{(e^{\lambda T}-1)^2} ds + \int_t^T \frac{e^{2\lambda(s-t)}}{(e^{\lambda T}-1)^2} ds,
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\lambda(e^{\lambda T}-1)^2}(e^{2\lambda T}-1), \\
&= \frac{1}{2\lambda(e^{\lambda T}-1)^2}(e^{\lambda T}-1)(e^{\lambda T}+1), \\
&= \frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}. \qquad (4.81)
\end{aligned}$$

The second integral gives

$$\begin{aligned}
\int_0^T e^{-4\tau}\eta^2 \ln[(u(s)-v(s))^2+1] ds &\leq e^{-4\tau}\eta^2 \ln[\|u-v\|^2+1] \times T, \\
&\leq e^{-4\tau}\eta^2 \ln[d_b(u,v)+1] \times T. \qquad (4.82)
\end{aligned}$$

Using (4.81) and (4.82) in (4.79), we get

$$\begin{aligned}
H_b(Tu,Tv) &\leq \sup_{t\in[a,b]} \Big(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\Big)^{\frac{1}{2}} \Big(e^{-4\tau}\eta^2 \ln[d_b(u,v)+1] \times T\Big)^{\frac{1}{2}}. \\
&\leq \Big(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\Big)^{\frac{1}{2}} e^{-2\tau}\eta \times \sqrt{T}\Big(\ln[d_b(u,v)+1]\Big)^{\frac{1}{2}}, \\
H_b(Tu,Tv)^2 &\leq \Big(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\Big) e^{-4\tau}\eta^2 \times T\Big(\ln[d_b(u,v)+1]\Big).
\end{aligned}$$

By (4.79) we have

$$\begin{aligned}
H_b(Tu,Tv)^2 &\leq e^{-4\tau}\Big(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\Big)\Big(\frac{2\lambda(e^{\lambda T}-1)}{T(e^{\lambda T}+1)}\Big) \times T\Big(\ln[d_b(u,v)+1]\Big), \\
H_b(Tu,Tv)^2 &\leq e^{-4\tau} \ln[d_b(u,v)+1], \\
H_b(Tu,Tv) &\leq e^{-2\tau}\sqrt{\ln[d_b(u,v)+1]}, \\
\Longrightarrow H_b(Tu,Tv) &\leq e^{-2\tau} M(u,v).
\end{aligned}$$

By passing to logarithms, we can write this as

$$\ln(H_b(Tu,Tv)) \leq (e^{-2\tau} M(u,v)). \qquad (4.83)$$

Similarly using the properties of F, ϕ, ψ and F_C in the above inequality we obtain

$$2\tau + F(\phi(sH_b(Tu,Tv))) \quad \leq \quad F(\phi(M(u,v))).$$

Now, for a function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(u) = \ln u$, for each $C(I,\mathbb{R})$ is a member of $\mathscr{F}$ and so it is amount to say that the operator T satisfies condition of Theorem 4.8 with $\eta, \beta, \gamma, \delta, \mu < 1$, and $L = 0$. Therefore, the conditions imposed in Theorem 4.11 are satisfied. Hence, $u \in C(I,\mathbb{R})$, that is; u is a solution of integral equation (4.77). Thus (4.74) has a solution.

Yan *et al.* (2012) and Harjani and Sadarangani (2010) proved that $\alpha(t) \leq T(\alpha(t))$, be a lower solution for (4.74) in partially ordered metric space. For b-metric space using this concept we can say that for $u < v$ all conditions of Theorem 4.11 are satisfied. Hence T has a unique fixed point.

4.6.3 Initial Valued Ordinary Differential Equation Problem in Partially Ordered b-Metric Spaces

As inspired by Qawaqneh *et al.* (2019), this section investigates the solution to the initial value problem in partially ordered b-metric space.

Consider the space $C(I)$, the class of real-valued continuous functions defined on $I = [a,b]$ endowed with the metric d_b given by

$$d_b(u,v) = \sup_{t \in I}\Big\{|u(t) - v(t)|^2 : t \in I\Big\},$$

for all $u, v \in C(I)$.

Note that $(C(I), d_b, s)$ is a complete partially ordered b-metric space. Furthermore, the partial order $C(I)$ is defined by

$$u, v \in C(I), u \leq v \implies u(t) \leq v(t),$$

for $t \in I$.

Consider the initial value problem

$$\begin{cases} u^{'}(t) = f(t,u(t)),\ t \in [a,b], \\ u(a) = u(0) = u_0, \end{cases} \tag{4.84}$$

where $I=[0,1]\subset\mathbb{R}^n,\ f:[0,1]\times\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function and u_0 is a real constant. We consider a space $C(I)(I=[0,1])$ of continuous function defined on I.

Theorem 4.12 Suppose that the problem (4.84) with $f:I\times\mathbb{R}^n\to\mathbb{R}$ continuous, and assume that there exists $\tau>0$ with

$$|f(s,u(s))-f(s,v(s))| \leq e^{-\tau}M(u,v), \tag{4.85}$$

where

$$\begin{aligned} M(u,v) &= \frac{\eta d_b(u,Tu).d_b(v,Tv)}{d_b(u,v)}+\mu d_b(u,v)+\beta d_b(u,Tu)+ \\ &\quad \gamma d_b(v,Tv)+\delta d_b(u,Tv)+Ld_b(v,Tu), \end{aligned}$$

such that for $u,v\in\mathbb{R}^n$ with $u>v$. Then the intial valued problem (4.84) has a unique solution on the interval I.

Proof. We define a map $T:C(I,\mathbb{R}^n)\to C(I,\mathbb{R}^n)$, for any $u,v\in\mathbb{R}^n$ by

$$Tu(t) = u_0+\int_0^1 f(s,u(s))ds, \tag{4.86}$$

for all $t\in I$.

$$\begin{aligned} H_b(Tu,Tv) &= \Big(\sup_{t\in I}|Tu(t)-Tv(t)|\Big)^2, \\ &\leq \left(\sup_{t\in I}\int_a^t |f(s,u(s)-f(s,v(s)|ds\right)^2, \\ &\leq \left(e^{-\tau}|u-v|\Big(\sup_{t\in I}\int_0^1 ds\Big)\right)^2, \\ &\leq \Big(e^{-\tau}|u-v|\Big(t-0\Big)\Big)^2, \\ &\leq e^{-2\tau}|u-v|^2(1-0)^2, \\ &\leq e^{-2\tau}M(u,v), \end{aligned} \tag{4.87}$$

which contradicts the condition imposed in Theorem 4.12. If we take $\tau>0$ such that $e^{-2\tau}<1$ and T is a contraction mappings. Then by Theorem 4.8, T has a unique fixed point which is a unique solution of the initial valued problem (4.84). Hence by following the similar procedure as from the above inequality, one can show that the conditions of Theorem 4.8 are satisfied.

□

CHAPTER FIVE

FIXED POINT THEOREMS FOR HYBRID MAPPINGS

5.1 Introduction

Hybrid fixed point theory combines two mappings; single and multivalued mappings. The method of hybrid fixed point can be used to derive another classical fixed point theorem results. The concept of hybrid pair of mapping is very consequential for the theory of fixed point and it has an important role in game theory, differential equation and optimization. Naimpally *et al.* (1986) generalised Goebel (1968) result in a hybrid of single-valued and multivalued maps satisfying a contractive condition.

Mustafa and Sims (2006, 2009) gave a generalisation of D-metric space to G-metric space soon after identifying some shortcomings concerning the fundamental topological structure on D-metric spaces. They defined several notions, such as continuity, completeness, compactness, convergence, and space product in the G-metric space setting. In doing so, they replaced the tetrahedral inequality with an inequality involving the repetition of indices. Later, Kaewcharoen *et al.* (2011) established the multi-valued notion in G-metric space. We recall these concepts for G metric spaces in subsection 2.3.6 and subsection 2.4.3.

Further, Sintunavarat and Kumam (2011) initiated the concept of common limit range (CLR) property in order to exhibit its sharpness over the (E.A) property due to Aamri and El Moutawakil (2002). Similarly, Chauhan *et al.* (2014) proved unified common fixed point theorems for a hybrid pair of mappings via an implicit relation involving altering distance function. Imdad *et al.* (2014) introduced the notion of common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric (semi-metric) space. Besides this, Imdad *et al.* (2015) established the joint common limit range notion and proved the common fixed point theorem for a pair of non-self mappings in metric space. Recently, Nashine *et al.* (2018) established a proof on common fixed point theorem for hybrid generalised (F,ψ)-contraction under common limit range property in metric spaces.

Beg and Pathak (2018) proved Nadler's theorem on weak partial metric spaces with application to homotopy result. Later, Kanwal *et al.* (2019) defined the notion of weak partial b-metric spaces and weak partial Hausdorff b-metric spaces along with the topology of weak partial b-metric space. Moreover, they generalised Nadler's theorem using weak partial Hausdorff b-metric space in the context of a weak partial b-metric space. We refer the reader for more information in subsection 2.3.11 and subsection 2.4.5.

On the other hand, Aserkar and Gandhi (2020) gave the results in b-metric space for weakly compatible mappings in pairs that satisfy the common limit range property. Karapinar *et al.* (2020) proved their results on p-hybrid Wardowski contraction for self map in metric space. Further, Mustafa *et al.* (2012) proved their results on common fixed points in G-metric spaces using the concept of $(E.A)$-property.

This chapter aims to prove the results for hybrid mapping as defined in section 2.5. The results are divided into five sections. Section 5.2 gives the definitions and preliminary results. In section 5.3, we establish the proof of common fixed point for two hybrid pairs of coincidentally idempotent non-self mappings in weakly partial b-metric space, which satisfies joint common limit range property (JCLR) in a generalised (F,ξ,η)-contraction mapping. Motivated by the results obtained by Wardowski and Van Dung (2014), Imdad *et al.* (2015), Secelean (2016), Joshi *et al.* (2017), Nashine et al. (2018), Nashine *et al.* (2020) and Aserkar and Gandhi (2020), we provide an illustrative example to support the theorem proved.

In section 5.4, we extend and prove the results for hybrid mapping via common limit range property in G-metric space setting. The theorem proved in this section will generalise the results due to Mustafa *et al.* (2012) from metric space setting to G-metric spaces. We also provide an illustrative example to validate the results.

In section 5.5, we prove the results for p-hybrid mappings via common limit range

(CLR) property in G-metric space setting. The theorems proved here will generalise the results due to Karapinar *et al.* (2013), Karapinar (2009), Nashine *et al.* (2018), Karapinar *et al.* (2019) and Karapinar *et al.* (2020) from metric space and quasi partial metric notion to G-metric space.

Finally, section 5.6 discusses an application for non-linear hybrid differential equation (HDE), which illustrate Theorem 5.4. In general, the results generalise and improve several known works of the existing literature.

5.2 Preliminaries

We will require the following preliminary definitions and theorems for establishing our results.

Motivated by Wardowski and Van Dung (2014), we introduce the notion of F-weak partial b-metric space as follows.

Definition 5.1 *Let (M, ρ_b) be a weak partial b-metric space. A map $T : M \to M$ is said to be an F-weak contraction on (M, ρ_b) if there exists $F \in \mathscr{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $\rho_b(Tx, Ty) > 0$, the following condition holds:*

$$\begin{aligned} \tau + F(\rho_b(Tx, Ty)) \quad \leq \quad & F\Big(\max\Big\{\rho_b(x,y), \rho_b(x,Tx), \rho_b(y,Ty), \\ & \frac{\rho_b(x,Ty)+\rho_b(y,Tx)}{2}\Big\}\Big). \end{aligned}$$

Motivated by Piri and Rahrovi (2015), we establish the concept of multivalued F-weak partial b-metric space that states below.

Definition 5.2 *Let (M, ρ_b) be a weak partial b-metric space. A map $T : M \to CB^{\rho_b}(M)$ is said to be multivalued F-weak contraction on (M, ρ_b) if there exists $F \in \mathscr{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $\mathscr{H}^{+}_{\rho_b}(Tx, Ty) > 0$, the following holds:*

$$\tau + F(\mathscr{H}^{+}_{\rho_b}(Tx, Ty)) \quad \leq \quad F(N(x,y)),$$

where,

$$N(x,y) \quad = \quad \max\left\{\rho_b(x,y), \rho_b(x,Tx), \rho_b(y,Ty), \frac{\rho_b(x,Ty)+\rho_b(y,Tx)}{2}\right\}.$$

Khan *et al.* (1984) established an altering distances concept between the points in metric space as follows:

Definition 5.3 *(Khan et al. 1984) A function $\xi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties satisfies:*

(i) *ξ is continuous and non-decreasing.*

(ii) *$\xi(t) = 0$ if and only if $t = 0$.*

Let $\xi, \eta \in \Psi$, Ψ denote the set of all continuous functions $\xi, \eta : [0,+\infty) \to [0,+\infty)$ with $s \geq 1$ a given real number such that

(iii) $s\xi(t) \leq \xi(t) - \eta(t)$ if and only if $t = 0$.

Imdad *et al.* (2015) established the concept of joint common limit range property for two hybrid pairs of non-self mappings as follows:

Definition 5.4 Let (X,d) be a metric space whereas Y an arbitrary non-empty set with $F,G : Y \to CB(X)$ and $f,g : Y \to X$. Then the pairs of hybrid mappings (F,f) and (G,g) are said to have the $(JCLR)$ property, if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A,B \in CB(X)$ such that

$$\begin{aligned}
\lim_{n\to\infty} Fx_n &= A, \ \lim_{n\to\infty} Gy_n = B, \\
\lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gy_n = t \in A \cap B \cap f(Y) \cap g(Y),
\end{aligned}$$

i.e., there exist $u,v \in Y$ such that $t = fu = gv \in A \cap B$.

Imdad *et al.* (2001) defined that a map is said to be coincidentally idempotent if it satisfies the condition given in the following definition:

Definition 5.5 *(Imdad* et al. *2001) Let (X,d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to CB(X)$ and $g : Y \to X$. The mapping g is said to be a coincidentally idempotent with respect to the mapping T, if $gx \in Tx$ with $gx \in Y$ imply $ggx = gx$ that is, g is idempotent at coincidence point of the pair (T,g).*

Aserkar and Gandhi (2020) gave the following result in b-metric space for weakly compatible mappings in pairs that satisfy the common limit range property.

The theorem of Aserkar and Gandhi (2020) is as follows:

Theorem 5.1 *(Aserkar and Gandhi 2020)* Let (X,d) be a b-metric space with $s \geq 1$ and $F,G,P,Q : X \to X$. Suppose that $\xi, \eta \in \psi$ and $L \geq 0$ such that

(i) (F,Q) satisfies CLR_P and (G,P) satisfies CLR_Q.

(ii) $s\xi(d(Fx,Gy)) \leq \xi(N_1(x,y)) - \eta(N_1(x,y)) + LN_2(x,y)$, where

$$\begin{aligned} N_1(x,y) &= \max\Bigg\{ d(Py,Qx), \frac{d(Qx,Fx)*d(Py,Gy)}{1+d(Fx,Gy)}, \\ & \frac{(d(Py,Fx))^2+(d(Qx,Gy))^2}{d(Py,Fx)+d(Qx,Gy)}, \\ & \frac{d(Qx,Fx)*d(Qx,Gy)+d(Py,Gy)*d(Py,Fx)}{d(Qx,Gy)+d(Py,Fx)} \Bigg\}, \end{aligned}$$

and

$$N_2(x,y) = \min\Big\{d(Qx,Fx), d(Qx,Gy), d(Py,Fx), d(Py,Gy)\Big\}, \quad (5.1)$$

for all $x,y \in X$.

(iii) The pair (F,Q) and (G,P) are weakly compatible.

Then F,G,P,Q have a unique common fixed point.

We introduce some proposition, definitions and theorems related to a common fixed point result using (CLR_f) in G-metric space.

Definition 5.6 (Kaewcharoen and Kaewkhao 2011) Let X be a non-empty set. Assume $f : X \to X$ and $T : X \to 2^X$ are two mappings. If $w = fx \in Tx$ for some $x \in X$, then x is called a coincidence point of the pair (T, f) and w is a point of coincidence of f and T. The mapping f and T are called weakly compatible if $fx \in Tx$ for some $x \in X$ implies $fTx \subseteq Tfx$.

Proposition 5.1 (Kaewcharoen and Kaewkhao 2011) Let X be a non-empty set. Assume $f : X \to X$ and $T : X \to 2^X$ are weakly compatible mappings. If f and T have a unique point of coincidence $u = fx \in Tx$, then u is a unique common fixed point of f and T.

Imdad *et al.* (2014), established the concept of common limit range property for a pair of hybrid mappings as follows:

Definition 5.7 (Imdad *et al.* 2014) Let (X,d) be a metric space with $f : X \to X$ and $T : X \to CB(X)$ be single and multivalued mappings. Then the pair of hybrid mappings (f,T) is said to have the (CLR) property, if there exists a sequence $\{x_n\}$ in X and $A \in 2^X$ such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n,$$

for some $u \in X$ and $A \in CB(X)$.

Mustafa *et al.* (2012) defined a concept of 0-mappings on G-metric spaces as follows:

Definition 5.8 (Mustafa *et al.* 2012) Let (X,d) be a G- metric space and let $T : X \to X$ be a mapping. For $A \subset X$, let $\delta(A) = \sup\{G(a,b,c), a,b,c \in A\}$ and $\forall x,y \in X$, define,

$$\begin{aligned} 0(x,T,n) &= \{x,Tx,T^2x,\dots T^nx\}, \\ 0(x,T,\infty) &= \{x,Tx,T^2x,T^3x\dots\}. \end{aligned}$$

Definition 5.9 (Mustafa *et al.* 2012) Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of elements of X, then for i, j, let

$$\begin{aligned} 0(x_i,j) &= \{x_i,x_{i+1},x_{i+2},\dots x_{i+j}\}, \\ 0(x_i,\infty) &= \{x_i,x_{i+1},x_{i+2},\dots\}. \end{aligned}$$

Mustafa *et al.* (2012) proved their results on common fixed points in G-metric spaces using the concept of (E.A) property as follows:

Theorem 5.2 (Mustafa *et al.* 2012) Let (X,G) be a complete G-metric space and suppose mapping $f,g : X \to X$ satisfy the following conditions:

(i) f and g are G-weakly commuting of type G_f,

(ii) $f(X) \subseteq g(X)$,

(iii) $g(X)$ is a G-complete subspace of X,

(iv) $G(fx,fy,fz) \leq \phi(M(x,y,z))$, for all $x,y,z \in X$,

where

$$M(x,y,z) \leq \left\{ \begin{array}{c} G(gx,gy,gz), G(gx,fy,gx) \\ G(gy,fx,gy), G(gz,fx,gz) \\ G(gz,fy,gz), G(gy,gz,gy), \\ G(gx,fz,gx). \end{array} \right\}.$$

If there exists $x_0 \in X$ such that $\delta(0(x_0,f,\infty)) < \infty$, then f and g have a unique common fixed point.

The notion of an almost altering distance function was introduced by Popa (1999).

Definition 5.10 (Popa 1999) A function $\psi : [0,\infty) \to [0,\infty)$ is almost altering distance if

(Ψ1) ψ is continuous;

(Ψ2) $\psi(t) = 0$ if and only if $t = 0$.

Example 5.1 (Popa 1999)

$$\psi(t) = \begin{cases} t, & \text{for } t = [0,1]; \\ \\ \frac{1}{t} & \text{for } t \in (1,\infty). \end{cases}$$

Karapinar *et al.* (2020) proved their results on p-hybrid Wardowski contraction as follows:

Definition 5.11 (Karapinar *et al.* 2020) A mapping $\mathcal{J} : (\mathcal{M},d) \to (\mathcal{M},d)$ is called a p- hybrid Wardowski contraction, if there is $G \in \beta$ such that

$$d(\mathcal{J}v,\mathcal{J}w) > 0 \Rightarrow \tau + G(d(\mathcal{J}v,\mathcal{J}w)) \quad \leq \quad G(\mathcal{A}^p_{\mathcal{J}}(v,w)),$$

for all $p > 0$. In particular, if the above inequality holds for $p = 0$, we say the $\mathcal{J}$ is a 0-hybrid Wardowski contraction.

Definition 5.12 (Karapinar 2019, Karapinar *et al.* 2020) Let $(\mathcal{M},d)$ be a metric space and $\mathcal{J}$ be a self-mapping on this space for $p \geq 0$ and $k_i \geq 0, i = 1,2,3,4$, such that $\sum_{i=1}^{4} k_i = 1$, we define the following expression.

$$\mathcal{A}^p_{\mathcal{J}}(v,w) \quad = \quad \begin{cases} \Big[k_1(d(v,w))^p + k_2(d(v,\mathcal{J}v))^p + k_3(d(w,\mathcal{J}w))^p \\ \\ +k_4\Big(\frac{d(v,\mathcal{J}v)+d(v,\mathcal{J}w)}{2}\Big)^p\Big]^{\frac{1}{p}}, \text{ for } p \geq 0, v,w \in \mathcal{M}; \\ \\ \Big[d(v,w)\Big]^{K_1}\Big[d(v,\mathcal{J}v)\Big]^{K_2}\Big[d(w,\mathcal{J}w)\Big]^{K_3}\Big[\frac{d(v,\mathcal{J}w)+d(w,\mathcal{J}v)}{2}\Big]^{K_4}, \\ \\ \text{for } p = 0, v,w \in \mathcal{M}\setminus\mathcal{J}(\mathcal{M}). \end{cases}$$

Theorem 5.3 [Theorem 4 and 5] (Karapinar *et al.* 2020)

(1) A p-hybrid Wardowski contraction self mapping on a complete metric space admits exactly one fixed point in $\mathcal{M}$.

(2) A 0-hybrid Wardowski contraction self mapping on a complete metric space admits fixed point in $\mathcal{M}$ provided that for each sequence $\{n_n\}$ in $(0,\infty)$, $\lim_{n\to\infty} n_n = 0$ iff $\lim_{n\to\infty} G(n_n) = 0$.

5.3 Common Fixed Point Results for a Hybrid Pair of Non-self Mappings via $(JCLR)$-Property in Weak Partial b-Metric Space

We commence by extending. Definition 5.4 to weak partial b-metric space for non-self mappings as follows:

Definition 5.13 *Let* (M,ρ_b) *be a weak partial b-metric space with* $f,g: X\to M$ *and* $G,T: X\to CB^{\rho_b}(M)$. *Then the pairs of hybrid mappings* (G,f) *and* (T,g) *are said to have joint common limit range property, denoted by* $(JCLR)$*-property, if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *and* $A,B\in CB^{\rho_b}(M)$ *such that*

$$\begin{aligned}\lim_{n\to\infty} Gx_n &= A, \lim_{n\to\infty} Ty_n = B,\\ \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gy_n = t,\end{aligned}$$

with $t\in f(X)\cap g(X)\cap A\cap B$, *that is, there exist* $u,v\in X$ *such that* $t=fu=gv\in A\cap B$.

Next, we extend Definition 5.5 to weak partial b-metric space as follows:

Definition 5.14 *Let* (M,ρ_b) *be a weak partial b-metric space with* $f: X\to M$ *and* $G: X\to CB^{\rho_b}(M)$. *The mapping* f *is said to be a coincidentally idempotent with respect to the mapping G, if* $u\in M$, $fu\in Gu$ *with* $fu\in M$ *imply* $ffu=fu$ *that is,* f *is idempotent at coincidence point of the pair* (G,f).

Now, we prove the following theorem which is an extended version of Theorem 5.1 in weak partial b-metric space for two hybrid pairs of non-self mappings, which satisfies joint common limit range property.

Theorem 5.4 *Let* $f,g: X\to M$ *be two self mappings of a weak partial b-metric space* (M,ρ_b) *with* $s\geq 1$ *and* $G,T: X\to CB^{\rho_b}(M)$ *be two multivalued mappings from* X *into* $CB^{\rho_b}(M)$. *Assume that* $\xi,\eta\in\psi$ *and* $L\geq 0$ *such that*

(i) *the hybrid pair* (G,f) *and* (T,g) *satisfies JCLR property,*

(ii) *there exists* $\tau>0$ *with* $\mathscr{H}^{+}_{\rho_b}(Gx,Ty)>0$ *such that*

$$\tau+F(s\xi(\mathscr{H}^{+}_{\rho_b}(Gx,Ty)))\leq F(\xi(N_1(x,y))-\eta(N_1(x,y))+LN_2(x,y)), \quad (5.2)$$

where

$$\begin{aligned}N_1(x,y) &= \max\Big\{\rho_b(gy,fx), \frac{\rho_b(fx,Gx)*\rho_b(gy,Ty)}{1+\rho_b(Gx,Ty)},\\ &\frac{(\rho_b(gy,Gx))^2+(\rho_b(fx,Ty))^2}{\rho_b(gy,Gx)+\rho_b(fx,Ty)},\\ &\frac{\rho_b(fx,Gx)*\rho_b(fx,Ty)+\rho_b(gy,Ty)*\rho_b(gy,Gx)}{\rho_b(fx,Ty)+\rho_b(gy,Gx)}\Big\},\end{aligned}$$

and

$$N_2(x,y)=\min\Big\{\rho_b(fx,Gx),\rho_b(fx,Ty),\rho_b(gy,Gx),\rho_b(gy,Ty)\Big\},$$

for all $x,y\in M$,

(iii) if $X \subset M$ and the pairs (G,f) and (T,g) are coincidentally commuting and coincidentally idempotent.

Then the pair (G,f) and (T,g) have a common fixed point in $u \in M$ and $\rho_b(u,u) = 0$.

Proof. Since the hybrid pairs (G,f) and (T,g) satisfies the *JCLR* property, by Definition 5.13 there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A,B \in CB^{\rho_b}(M)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Gx_n, \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Ty_n,$$

for some $u,v \in X$ and $t = fv = gu \in A \cap B$. We assert that $gu \in Tu$. If not, then using $x = x_n$ and $y = u$ in (5.2), we get

$$\tau + F(s\xi(\mathscr{H}^+_{\rho_b}(Gx_n,Tu))) \le F(\xi(N_1(x_n,u)) - \eta(N_1(x_n,u)) + LN_2(x_n,u)), \quad (5.3)$$

where

$$\begin{aligned} N_1(x_n,u)) \;=\; \max\Bigg\{ & \rho_b(gu,fx_n), \frac{\rho_b(fx_n,Gx_n)*\rho_b(gu,Tu)}{1+\rho_b(Gx_n,Tu)}, \\ & \frac{(\rho_b(gu,Gx_n))^2+(\rho_b(fx_n,Tu))^2}{\rho_b(gu,Gx_n)+\rho_b(fx_n,Tu)}, \\ & \frac{\rho_b(fx_n,Gx_n)*\rho_b(fx_n,Tu)+\rho_b(gu,Tu)*\rho_b(gu,Gx_n)}{\rho_b(fx_n,Tu)+\rho_b(gu,Gx_n)} \Bigg\}, \end{aligned} \quad (5.4)$$

Taking limit as $n \to \infty$ in (5.4), we get

$$\begin{aligned} \le\; \max\Bigg\{ & \rho_b(gu,gu), \frac{\rho_b(gu,A)*\rho_b(gu,Tu)}{1+\rho_b(A,Tu)}, \\ & \frac{(\rho_b(gu,A))^2+(\rho_b(gu,Tu))^2}{\rho_b(gu,A)+\rho_b(gu,Tu)}, \\ & \frac{\rho_b(gu,A)*\rho_b(gu,Tu)+\rho_b(gu,Tu)*\rho_b(gu,A)}{\rho_b(gu,Tu)+\rho_b(gu,A)} \Bigg\}, \\ \le\; \max\Bigg\{ & \rho_b(t,t), \frac{\rho_b(t,A)*\rho_b(gu,Tu)}{1+\rho_b(A,Tu)}, \\ & \frac{(\rho_b(t,A))^2+(\rho_b(gu,Tu))^2}{\rho_b(t,A)+\rho_b(gu,Tu)}, \\ & \frac{\rho_b(t,A)*\rho_b(gu,Tu)+\rho_b(gu,Tu)*\rho_b(t,A)}{\rho_b(gu,Tu)+\rho_b(t,A)} \Bigg\}, \end{aligned} \quad (5.5)$$

using Definition 2.29 and (2.21) in (5.5), we get

$$\begin{aligned} \le\; \max\Bigg\{ & 0, \frac{0*\rho_b(gu,Tu)}{1+\rho_b(A,Tu)}, \frac{(0)^2+(\rho_b(gu,Tu))^2}{0+\rho_b(gu,Tu)}, \\ & \frac{0*\rho_b(gu,Tu)+\rho_b(gu,Tu)*0}{\rho_b(gu,Tu)+0} \Bigg\}, \end{aligned}$$

$$\begin{aligned}
&\leq \max\left\{0,0,\frac{\rho_b(gu,Tu))^2}{\rho_b(gu,Tu)},0\right\},\\
&\leq \max\Big\{0,0,\rho_b(gu,Tu),0\Big\},\\
&= \rho_b(gu,Tu). \qquad (5.6)
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
N_2(x_n,u)) &= \min\Big\{\rho_b(fx_n,Gx_n),\rho_b(fx_n,Tu),\rho_b(gu,Gx_n),\rho_b(gu,Tu)\Big\},\\
&\leq \min\Big\{\rho_b(gu,A),\rho_b(gu,Tu),\rho_b(gu,A),\rho_b(gu,Tu)\Big\},\\
&\leq \min\Big\{\rho_b(t,A),\rho_b(gu,Tu),\rho_b(t,A),\rho_b(gu,Tu)\Big\},
\end{aligned}$$

$$\begin{aligned}
&\leq \min\Big\{0,\rho_b(gu,Tu),0,\rho_b(gu,Tu)\Big\}\\
&= 0. \qquad (5.7)
\end{aligned}$$

Using (5.7) and (5.6) in (5.3), one obtains

$$\begin{aligned}
&\tau+F(s\xi\mathscr{H}^+_{\rho_b}(A,Tu))\leq F(\xi\rho_b(gu,Tu)-\eta\rho_b(gu,Tu)+L(0)),\\
&\tau+F(s\xi\mathscr{H}^+_{\rho_b}(A,Tu))\leq F(\xi\rho_b(gu,Tu)-\eta\rho_b(gu,Tu)).
\end{aligned}$$

Since $\tau>0$, in viewing the properties of η,ξ, and F is strictly increasing, by $(F1)$ we have

$$\begin{aligned}
\mathscr{H}^+_{\rho_b}(A,Tu)&<\rho_b(gu,Tu)\\
s\xi\mathscr{H}^+_{\rho_b}(A,Tu)&\leq(\xi-\eta)\rho_b(gu,Tu)
\end{aligned}$$

As $t=fv=gu\in A\cap B$, it follows that

$$\mathscr{H}^+_{\rho_b}(A,Tu))\leq\frac{\xi-\eta}{s\xi}\Big\{\rho_b(gu,Tu)\Big\}.$$

Thus,

$$\rho_b(gu,Tu)<\mathscr{H}^+_{\rho_b}(A,Tu)<\frac{\xi-\eta}{s\xi}\Big\{\rho_b(gu,Tu)\Big\},$$

a contradiction. Hence $gu\in Tu$ which shows that the pair (T,g) has a coincidence point u in M.

Similarly, we assert that $fv\in Gv$. Suppose that $fv\neq Gv$, then using $x=v$ and $y=y_n$ in (5.2), one gets

$$\tau+F(s\xi(\mathscr{H}^+_{\rho_b}(Gv,Ty_n)))\leq F(\xi(N_1(v,y_n))-\eta(N_1(v,y_n))+LN_2(v,y_n,)), \qquad (5.8)$$

where

$$\begin{aligned} N_1(v,y_n)) &= \max\Bigg\{\rho_b(gy_n,fv), \frac{\rho_b(fv,Gv)*\rho_b(gy_n,Ty_n)}{1+\rho_b(Gv,Ty_n)}, \\ &\quad \frac{(\rho_b(gy_n,Gv))^2+(\rho_b(fv,Ty_n))^2}{\rho_b(gy_n,Gv)+\rho_b(fv,Ty_n)}, \\ &\quad \frac{\rho_b(fv,Gv)*\rho_b(fv,Ty_n)+\rho_b(gy_n,Ty_n)*\rho_b(gy_n,Gv)}{\rho_b(fv,Ty_n)+\rho_b(gy_n,Gv)}\Bigg\}, \end{aligned} \tag{5.9}$$

Taking limit as $n \to \infty$ in (5.9), we have

$$\begin{aligned} &\leq \max\Bigg\{\rho_b(fv,fv), \frac{\rho_b(fv,Gv)*\rho_b(fv,B)}{1+\rho_b(Gv,B)}, \\ &\quad \frac{(\rho_b(fv,Gv))^2+(\rho_b(fv,B))^2}{\rho_b(fv,Gv)+\rho_b(fv,B)}, \\ &\quad \frac{\rho_b(fv,Gv)*\rho_b(fv,B)+\rho_b(fv,B)*\rho_b(fv,Gv)}{\rho_b(fv,B)+\rho_b(fv,Gv)}\Bigg\}, \end{aligned}$$

by definition (5.13), we have

$$\begin{aligned} &\leq \max\Bigg\{\rho_b(t,t), \frac{\rho_b(fv,Gv)*\rho_b(t,B)}{1+\rho_b(Gv,B)}, \\ &\quad \frac{(\rho_b(fv,Gv))^2+(\rho_b(t,B))^2}{\rho_b(fv,Gv)+\rho_b(t,B)}, \\ &\quad \frac{\rho_b(fv,Gv)*\rho_b(t,B)+\rho_b(t,B)*\rho_b(fv,Gv)}{\rho_b(t,B)+\rho_b(fv,Gv)}\Bigg\}, \end{aligned} \tag{5.10}$$

using Definition 2.29 and (2.21) in (5.10), we get

$$\begin{aligned} &\leq \max\Bigg\{0, \frac{\rho_b(fv,Gv)*0}{1+\rho_b(Gv,B)}, \frac{(\rho_b(fv,Gv))^2+(0)^2}{\rho_b(fv,Gv)+0}, \\ &\quad \frac{\rho_b(fv,Gv)*0+0*\rho_b(fv,Gv)}{0+\rho_b(fv,Gv)}\Bigg\}, \\ &\leq \max\Bigg\{0,0,\frac{\rho_b(fv,Gv))^2}{\rho_b(fv,Gv)},0\Bigg\}, \\ &\leq \max\Big\{0,0,\rho_b(fv,Gv),0\Big\}, \\ &= \rho_b(fv,Gv). \end{aligned} \tag{5.11}$$

Consequently, we have

$$\begin{aligned} N_2(v,y_n,)) &= \min\Big\{\rho_b(fv,Gv),\rho_b(fv,Ty_n),\rho_b(gy_n,Gv),\rho_b(gy_n,Ty_n)\Big\}, \\ &\leq \min\Big\{\rho_b(fv,Gv),\rho_b(fv,B),\rho_b(fv,Gv),\rho_b(fv,B)\Big\}, \\ &\leq \min\Big\{\rho_b(fv,Gv),\rho_b(t,B),\rho_b(fv,Gv),\rho_b(t,B)\Big\}, \\ &\leq \min\Big\{\rho_b(fv,Gv),0,\rho_b(fv,Gv),0\Big\} \\ &= 0. \end{aligned} \tag{5.12}$$

Using (5.12) and (5.11) in (5.8), one obtains

$$\begin{gathered} \tau+F(s\xi\mathscr{H}^+_{\rho_b}(Gv,B)) \leq F(\xi\rho_b(fv,Gv)-\eta\rho_b(fv,Gv)+L(0)), \\ \tau+F(s\xi\mathscr{H}^+_{\rho_b}(Gv,B)) \leq F(\xi\rho_b(fv,Gv)-\eta\rho_b(fv,Gv)). \end{gathered}$$

Since $\tau > 0$, in viewing the properties of η, ξ, and F is strictly increasing, by $(F1)$ we have

$$\begin{gathered} \mathscr{H}^+_{\rho_b}(Gv,B) < \rho_b(fv,Gv) \\ s\xi\mathscr{H}^+_{\rho_b}(Gv,B) \leq (\xi-\eta)\rho_b(fv,Gv) \end{gathered}$$

As $t = fv = gu \in A\cap B$, it follows that

$$\mathscr{H}^+_{\rho_b}(Gv,B)) \leq \frac{\xi-\eta}{s\xi}\Big\{\rho_b(fv,Gv)\Big\}.$$

Thus,

$$\rho_b(fv,Gv) < \mathscr{H}^+_{\rho_b}(Gv,B) < \frac{\xi-\eta}{s\xi}\Big\{\rho_b(fv,Gv)\Big\},$$

a contradiction. Hence $fv \in Gv$ which shows that the pair (G,f) has a coincidence point v in M.

Next, we show that $gu \in Tu$ and $fv \in Gv$, if not, then using $x = u$ and $y = v$ in (5.2), we get

$$\tau+F(s\xi(\mathscr{H}^+_{\rho_b}(Gu,Tv))) \leq F(\xi(N_1(u,v))-\eta(N_1(u,v))+LN_2(u,v)), \tag{5.13}$$

where

$$\begin{aligned} N_1(u,v)) = \max\Bigg\{&\rho_b(gv,fu),\frac{\rho_b(fu,Gu)*\rho_b(gv,Tu)}{1+\rho_b(Gu,Tv)}, \\ &\frac{(\rho_b(gv,Gu))^2+(\rho_b(fu,Tv))^2}{\rho_b(gv,Gu)+\rho_b(fu,Tv)}, \\ &\frac{\rho_b(fu,Gu)*\rho_b(fu,Tv)+\rho_b(gv,Tv)*\rho_b(gv,Gu)}{\rho_b(fu,Tv)+\rho_b(gv,Gu)}\Bigg\}, \end{aligned}$$

using (2.21) in the above equation, one obtains

$$\begin{aligned} &\leq \max\left\{ \rho_b(gv, fu), 0, 0, 0 \right\}, \\ &= \rho_b(gv, fu). \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} N_2(u,v) &= \min\Big\{\rho_b(fu,Gu), \rho_b(fu,Tv), \rho_b(gv,Gu), \rho_b(gv,Tv)\Big\}, \\ &\leq \min\Big\{0,0,0,0\Big\}, \\ &= 0. \end{aligned} \tag{5.15}$$

Using (5.15) and (5.14) in (5.13), one gets

$$\begin{aligned} \tau + F(s\xi\mathscr{H}^+_{\rho_b}(Gu,Tv)) &\leq F(\xi\rho_b(gv,fu)) - \eta\rho_b(gv,fu) + L(0)), \qquad (5.16) \\ \tau + F(s\xi\mathscr{H}^+_{\rho_b}(Gu,Tv)) &\leq F(\xi\rho_b(gv,fu)) - \eta\rho_b(gv,fu), \end{aligned}$$

In viewing the properties of τ, η, ξ, and F is strictly increasing, by $(F1)$ and definition 5.3, we have

$$\begin{aligned} \mathscr{H}^+_{\rho_b}(Gu,Tv) &\leq \rho_b(gv,fu) \\ \Longrightarrow s\xi\mathscr{H}^+_{\rho_b}(Gu,Tv) &\leq (\xi - \eta)\rho_b(gv,fu) \end{aligned}$$

As $t = fv = gu \in A \cap B$, it follows that

$$\mathscr{H}^+_{\rho_b}(Gu,Tv) \leq \frac{\xi - \eta}{s\xi}\rho_b(gv,fu).$$

Thus,

$$\rho_b(gv,fu) < \mathscr{H}^+_{\rho_b}(Gu,Tv) < \frac{\xi - \eta}{s\xi}\rho_b(gv,fu),$$

a contradiction. Hence $gu \in Tu$ and $fv \in Gv$ which shows that the pair (T,g), (G,f) has a coincidence point $u = v$ in M.

Suppose that $X \in M$. Since v is a coincidence point of the pair (G,f) which is coincidentally commuting and coincidentally idempotent. With respect to mapping G, we have $fv \in Gv$ and $ffv = fv$, therefore $fv = ffv \in f(Gv) \subset G(fv)$ which shows that fv is a common fixed point of the pair (G,f). Similarly, u is a coincidence point of the pair (T,g) which is coincidentally commuting and coincidentally idempotent concerning mapping T, one can easily show that gu is a common fixed point of the pair (T,g).

Moreover, if u and v are coincidence points which are coincidentally commuting and coincidentally idempotent, then there exists $u \in C(T,g)$ and $v \in C(G,f)$ such that $gu = Tu, fv = Gv$.

Hence $u = v = gu = fv$, consequently, u is a common fixed point of the two hybrid pairs of mappings (G,f) and (T,g) in M. □

Example 5.2 *Let $X=[0,2]\subset[0,\infty)=M$ be a weak partial b-metric space equipped with metric $\rho_b(x,y)=|x-y|^2+1$, for all $x,y\in M$. Let $G,T:X\to CB^{\rho_b}(M)$ be defined as*

$$Gx=\begin{cases}[\frac{3}{5},\frac{3}{2}], \ if \ \ 0\leq x\leq 1,\\ \\ [\frac{1}{4},\frac{1}{2}], \ if \ \ 0\leq x<2.\end{cases}$$

$$Tx=\begin{cases}[\frac{3}{2},2], \ if \ \ 0\leq x<1,\\ \\ [\frac{1}{2},2], \ if \ \ 1\leq x\leq 2.\end{cases}$$

Suppose $f,g:X\to M$ be defined as

$$fx=\begin{cases}1, \ if \ \ \ 0\leq x\leq 1,\\ \\ \frac{3x}{5}, \ if \ \ 1<x\leq 2.\end{cases}$$

$$gx=\begin{cases}\frac{3x}{2}, \ if \ \ 0\leq x<1,\\ \\ 1, \ if \ \ \ 1\leq x\leq 2.\end{cases}$$

Let $F:\mathbb{R}^+\to\mathbb{R}$ be defined by $F(a)=\ln a+a$ and $\xi,\psi:[0,\infty)\to[0,\infty)$ such that $\xi(t)=\frac{1}{10}t, \eta(t)=\frac{t+1}{2}, L=5$, $s=2$ and $\tau=1$, then, Equation 5.2 takes the form

$$\frac{s\xi(\mathscr{H}^+_{\rho_b}(Gx,Ty))}{\xi(N_1(x,y))-\eta(N_1(x,y))+LN_2(x,y)}e^{s\xi(\mathscr{H}^+_{\rho_b}(Gx,Ty))-[\xi(N_1(x,y))-\eta(N_1(x,y))+LN_2(x,y)]}$$
$$\leq e^{-\tau}. \tag{5.17}$$

Choosing two sequences $\{x_n\}=\{1-\frac{1}{2n}\}$ and $\{y_n\}=\{1+\frac{1}{2n}\}$ in X, one can see that the pairs (G,f) and (T,g) satisfy $(JCLR)$ property, i.e.

$$\begin{aligned}\lim_{n\to\infty}f\left\{1-\frac{1}{2n}\right\} &= 1\in\left[\frac{3}{5},\frac{3}{2}\right]=\lim_{n\to\infty}G\left\{1-\frac{1}{2n}\right\},\\ \lim_{n\to\infty}g\left\{1+\frac{1}{2n}\right\} &= 1\in\left[\frac{1}{2},\frac{3}{2}\right]=\lim_{n\to\infty}T\left\{1+\frac{1}{2n}\right\}.\end{aligned}$$

Now to verify condition (5.17) we distinguish the following cases;

Case I

For $x\in[0,1]$, $y\in[1,2]$ and applying Definition 2.33, we have

$$\begin{aligned}\mathscr{H}^+_{\rho_b}(Gx,Ty) &= \mathscr{H}^+_{\rho_b}\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right)\\ &= \frac{1}{2}\left\{\sup\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right)+\sup\left(\left[\frac{1}{2},2\right],\left[\frac{3}{5},\frac{3}{2}\right]\right)\right\}.\end{aligned}\tag{5.18}$$

$$\begin{aligned}\sup\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right) &= \max\left\{\rho_b\left(\frac{3}{5},\left[\frac{1}{2},2\right]\right),\rho_b\left(\frac{3}{2},\left[\frac{1}{2},2\right]\right)\right\}\\ &= \max\left\{1.01, 1.25\right\}\\ &= 1.25. \end{aligned}\tag{5.19}$$

$$\begin{aligned}\sup\left(\left[\frac{1}{2},2\right],\left[\frac{3}{5},\frac{3}{2}\right]\right) &= \max\left\{\rho_b\left(\frac{1}{2},\left[\frac{3}{5},\frac{3}{2}\right]\right),\rho_b\left(2,\left[\frac{3}{5},\frac{3}{2}\right]\right)\right\}\\ &= \max\left\{1.01, 1.25\right\}\\ &= 1.25. \end{aligned}\tag{5.20}$$

By applying (5.20) *and* (5.19) *in* (5.18) *we get*

$$\mathscr{H}^{+}_{\rho_b}(Tx,Gy) = 1.25.$$

Similarly, we calculate the following metric

$$\begin{aligned}\rho_b(gy,fx) &= \rho_b(1,1)=1,\\ \rho_b(fx,Gx) &= \rho_b\left(1,\left[\frac{3}{5},\frac{3}{2}\right]\right)=1.16,\\ \rho_b(gy,Ty) &= \rho_b\left(1,\left[\frac{1}{2},2\right]\right)=1.25,\\ \rho_b(Gx,Ty) &= \rho_b\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right)=1.25,\\ \rho_b(gy,Gx) &= \rho_b\left(1,\left[\frac{3}{5},\frac{3}{2}\right]\right)=1.16,\\ \rho_b(fx,Ty) &= \rho_b\left(1,\left[\frac{1}{2},2\right]\right)=1.25.\end{aligned}$$

It follows that,

$$\begin{aligned}N_1(x,y)=\max\Bigg\{1,\frac{1.16*1.25}{1+1.25},\frac{(1.16)^2+(1.25)^2}{1.16+1.25},\\ \frac{1.16*1.25+1.25*1.16}{1.25+1.16}\Bigg\}=1.207,\end{aligned}$$

and

$$N_2(x,y)=\min\left\{1.16,1.25,1.16,1.25\right\}=1.16.$$

Therefore, (5.17) *reduces to*

$$\begin{aligned}\frac{2\times 0.1\times 1.25}{0.1\times 1.207-1.1035+5\times 1.16}e^{2\times 0.1\times 1.25-[0.1\times 1.207-1.1035+5\times 1.16]} &\leq e^{-\tau},\\ \frac{0.25}{4.8172}e^{0.25-4.8172} &\leq e^{-1},\\ \frac{0.25}{4.5672}e^{-4.5672} &\leq e^{-1},\end{aligned}$$

which is true.

Case II *For* $x\in[1,2]$*,* $y\in[0,1]$ *and using Definition 2.33, we have*

$$\begin{aligned}\mathscr{H}^{+}_{\rho_b}(Gx,Ty) &= \mathscr{H}^{+}_{\rho_b}\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right)\\ &= \frac{1}{2}\left\{\sup\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right)+\sup\left(\left[\frac{3}{2},2\right],\left[\frac{1}{4},\frac{1}{2}\right]\right)\right\}. \quad (5.21)\end{aligned}$$

$$\begin{aligned}\sup\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right) &= \max\left\{\rho_b\left(\frac{1}{4},\left[\frac{3}{2},2\right]\right),\rho_b\left(\frac{1}{2},\left[\frac{3}{2},2\right]\right)\right\}\\ &= \max\left\{2.5625,2\right\}\\ &= 2.5625. \quad (5.22)\end{aligned}$$

$$\begin{aligned}\sup\left(\left[\frac{3}{2},2\right],\left[\frac{1}{4},\frac{1}{2}\right]\right) &= \max\left\{\rho_b\left(\frac{3}{2},\left[\frac{1}{4},\frac{1}{2}\right]\right),\rho_b\left(2,\left[\frac{1}{4},\frac{1}{2}\right]\right)\right\}\\ &= \max\left\{2,3.25\right\}\\ &= 3.25. \quad (5.23)\end{aligned}$$

By applying (5.23) *and* (5.22) *in* (5.21) *we get*

$$\mathscr{H}^{+}_{\rho_b}(Gx,Ty) = 2.90625,$$

Similarly, we calculate the following metric

$$\begin{aligned}\rho_b(gy,fx) &= \rho_b\left(0,\frac{3}{5}\right)=1.36,\\ \rho_b(fx,Gx) &= \rho_b\left(\frac{3}{5},\left[\frac{1}{4},\frac{1}{2}\right]\right)=1.01,\\ \rho_b(gy,Ty) &= \rho_b\left(0,\left[\frac{3}{2},2\right]\right)=3.25,\\ \rho_b(Gx,Ty) &= \rho_b\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right)=2.90625,\\ \rho_b(gy,Gx) &= \rho_b\left(0,\left[\frac{1}{4},\frac{1}{2}\right]\right)=1.0625,\\ \rho_b(fx,Ty) &= \rho_b\left(\frac{6}{5},\left[\frac{3}{2},2\right]\right)=1.09.\end{aligned}$$

It follows that,

$$N_1(x,y) = \max\Big\{1.36, \frac{1.01*3.25}{1+2.90625}, \frac{(1.0625)^2+(1.09)^2}{1.0625+1.09}, \\ \frac{1.01*1.09+3.25*1.0625}{1.09+1.0625}\Big\} = 2.115691057,$$

and

$$N_2(x,y) = \min\Big\{1.01, 1.09, 1.0625, 3.25\Big\} = 1.01.$$

Therefore, (5.17) reduces to

$$\frac{0.58125}{3.703723516} e^{0.58125-3.703723516} \leq e^{-\tau}.$$
$$\frac{0.58125}{3.703723516} e^{-3.122473516} \leq e^{-1}.$$

which is true.

Notice that for $x,y \in [0,1]$ and $x,y \in [1,2]$, Equation (5.17) is true. Thus, all conditions of Theorem 5.4 are satisfied, and the hybrid pairs (G,f) and (T,g) have the common fixed point in M. Consider $v = 1$ be a coincidence point of the pair (G,f), then we have

(1) $f1 = 1 \in G1 = [\frac{3}{5}, \frac{3}{2}]$,

(2) $ff1 = f1 = 1$,

(3) $f1 = ff1 \in f(G1) \subset G(f1)$ *and*

Similarly, if we consider $u = 1$ as a coincidence point of the pair (T,g), it is easily proved that $u = v = 1$ and 1 is a unique common fixed point for the two pairs of hybrid mappings (G,f) and (T,g).

5.4 Existence Results for Hybrid Mappings using (CLR_f)- Property in G-Metric Spaces

To establish the results in this section, we start by extending Definition 5.7 using the G–metric space concept.

Definition 5.15 Let (X,G) be a G-metric space with $f : X \to X$ and $T : X \to CB(X)$ be single and multivalued mappings. Then the pair (f,T) of hybrid mappings satisfy a common limit range (CLR_f) property with respect to f, if there exists a sequence $\{x_n\}$ in X and $A \in CB(X)$ such that

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n,$$

for some $u \in X$.

Now, we prove the following theorem which is an extension of Theorem 5.2.

Theorem 5.5 Let (X,G) be a complete G- metric space and suppose mapping $f,T : X \to 2^X$ with almost altering distance $\psi \in \Psi$ satisfy the following conditions:

(a) f and T are weakly compatible;

(b) f and T satisfy CLR_f property;

(c) $TX \subseteq f(X)$;

(d) $T(X)$ is a G-complete subspace of X;

(e) $H_G(Tx,Ty,Tz) \leq \psi(M(x,y,z))$; for all $x,y,z \in X$,

where

$$M(x,y,z) \quad \leq \quad \max\left\{\begin{array}{c} G(fx,fy,fz),G(fx,Ty,fx), \\ G(fy,Tx,fy),G(fz,Tx,fz), \\ G(fz,Ty,fz),G(fy,Tz,fy), \\ G(fx,Tz,fx) \end{array}\right\}. \tag{5.24}$$

Then f and T have a unique common fixed point.

Proof. From condition (a), using Definition 5.6 one can show that f and T are weakly compatible. Since $fx \in Tx \Rightarrow fTx \subseteq Tfx$. Thus $T(X) \subseteq f(X)$ and $T(X)$ is a G-complete subspace of X.

Applying Definition 5.15, as the pair (f,T) satisfy CLR_f property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n \quad = \quad fu \in A = \lim_{n\to\infty} Tx_n,$$

for some $u \in X$ and $A \in 2^X$.

First, we show that $fu \in Tu$. If $fu \neq Tu$, for all $u \in X$, using Equation 5.24 with $x = x_n$ and $y = u$, we get

$$H_G(Tx_n,Tu,Tz) \quad \leq \quad \psi(M(x_n,u,z)), \tag{5.25}$$

where

$$M(x_n,u,u) \quad \leq \quad \max\left\{\begin{array}{c} G(fx_n,fu,fz),G(fx_n,Tu,fx_n), \\ G(fu,Tx_n,fu),G(fz,Tx_n,fz), \\ G(fz,Tu,fz),G(fu,Tz,fu), \\ G(fx_n,Tz,fx_n) \end{array}\right\}. \tag{5.26}$$

By taking $y = z$ in (5.26) we have

$$M(x_n,u,u) \leq \max \left\{ \begin{array}{c} G(fx_n, fu, fu), G(fx_n, Tu, fx_n), \\ G(fu, Tx_n, fu), G(fu, Tx_n, fu), \\ G(fu, Tu, fu), G(fu, Tu, fu), \\ G(fx_n, Tu, fx_n), \end{array} \right\} \tag{5.27}$$

Passing to the limits as $n \to \infty$ in (5.27) with $fx_n = fu$, $A = Tx_n$, we obtain

$$\begin{aligned} M(x_n,u,u) &\leq \max \left\{ \begin{array}{c} G(fu, fu, fu), G(fu, A, fu), \\ G(fu, A, fu), G(fu, A, fu), \\ G(fu, A, fu), G(fu, A, fu), \\ G(fu, A, fu), \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} 0, G(fu, A, fu), \\ G(fu, A, fu), G(fu, A, fu), \\ G(fu, A, fu), G(fu, A, fu), \\ G(fu, A, fu), \end{array} \right\} \end{aligned}$$

$$\Rightarrow M(x_n,u,u) = G(fu, A, fu). \tag{5.28}$$

Using (5.28) in (5.25) as a result yields

$$H_G(A, Tu, Tu) \leq \psi(G(fu, A, fu)). \tag{5.29}$$

As ψ is non-decreasing we have

$$H_G(A, Tu, Tu) < G(fu, A, fu), \tag{5.30}$$

Using (5.30) and Lemma 2.16, we get

$$\begin{aligned} G(fu, Tu, Tu) &\leq G(fu, A, fu) + \varepsilon, \\ G(fu, Tu, Tu) - G(fu, A, fu) &\leq \varepsilon, \\ \Rightarrow 0 &< \varepsilon, \end{aligned} \tag{5.31}$$

which is a contradiction. Hence, $fu \in Tu$ which shows that the pair (f, T) has a coincidence point.

Next, assume that $fu \in Tu$ and $fv \in Tv$. By Lemma 2.15 and Proposition 5.1, we have

$$G(fu, v, v) < G(Tu, Tv, Tv).$$

Now, we prove the uniqueness of a point of coincidence of f and T. We prove that the fixed point of T is unique. Assume that $v \in X$ is another coincidence fixed point of f and T such that $v \neq u$. Then, since u and v are such that $fu \in Tu, fv \in Tv \in X$, we set $x = u$ and $y = v$ in (5.24) which yields

$$H_G(Tu, Tv, Tz) \leq \psi(M(u, v, z)), \tag{5.32}$$

where,

$$M(u,v,z) \quad \leq \quad \max\left\{\begin{array}{c} G(fu,fv,fz), G(fu,Tv,fu), \\ G(fv,Tu,fv), G(fz,Tu,fz), \\ G(fz,Tv,fz), G(fv,Tz,fv), \\ G(fu,Tz,fu) \end{array}\right\}. \tag{5.33}$$

By taking $y = z$ in (5.33) we have

$$M(u,v,v) \quad \leq \quad \max\left\{\begin{array}{c} G(fu,fv,fv), G(fu,Tv,fu), \\ G(fv,Tu,fv), G(fv,Tu,fv), \\ G(fv,Tv,fv), G(fv,Tv,fv), \\ G(fu,Tv,fu), \end{array}\right\}$$

$$\Rightarrow M(u,v,v) \quad = \quad G(fu,Tv,fu). \tag{5.34}$$

Using (5.34) in (5.32) as a result yields

$$H_G(Tu,Tv,Tv) \quad \leq \quad \psi(G(fu,Tv,fu)). \tag{5.35}$$

As ψ is non-decreasing we have,

$$H_G(Tu,Tv,Tv) \quad < \quad G(fu,Tv,fu), \tag{5.36}$$

which is a contradiction. Hence, $fv \in Tv$ which shows that the pair (f,T) has a coincidence point. Thus, (f,T) has a unique common fixed point which is u, that is, the assumption given in Theorem 5.5 is validated. □

Furthermore, we formulate the two corollaries:

Corollary 5.1 *Let (X,G) be a G-metric space. Assume that $f : X \to X$ and $T : X \to 2^X$ satisfy all the conditions imposed in Theorem 5.5 if any of the following contractions is applied:*

(i)

$$H_G(Tx,Ty,Ty) \quad = \quad k\max\left\{\begin{array}{c} G(fy,Ty,Ty)+ \\ \Big(G(fx,Ty,Ty), 2G(fy,Tx,Tx)\Big) \end{array}\right\} \tag{5.37}$$

for all $x,y,z \in X$, where $k \in [0,\frac{1}{3})$.

(ii)

$$H_G(Tx,Ty,Tz) = \left\{\begin{array}{c} G(fx,fy,fz), G(fx,Tx,Tx) \\ G(fy,Ty,Ty), G(fy,Tz,Tz) \\ G(fx,Ty,Ty), G(fx,Tz,Tz), \\ G(fz,Tx,Tx), \end{array}\right\} \tag{5.38}$$

(iii)

$$H_G(Tx,Ty,Tz) \;=\; \left\{\begin{array}{c}\alpha G(fx,fy,fz)+\\ \beta[G(fx,Tx,Tx)+G(fy,Ty,Ty)\\ +G(fz,Tz,Tz)]\end{array}\right\} \qquad (5.39)$$

for all $x,y,z \in X$ *and* $0 \leq \alpha + 3\beta < 1$.

(iv)

$$H_G(Tx,Ty,Tz) \;\leq\; \psi(M(x,y,z)),$$

where

$$M(x,y,z) \;=\; \max\left\{\begin{array}{c}G(fx,fy,fy),G(fx,Tx,Tx)\\ G(fy,Ty,Ty),G(fz,Tz,Tz),\\ \frac{G(fx,fy,Tx)}{2},\\ \frac{G(fx,Ty,Ty)+G(fx,Tz,Tz)+G(fy,Tx,Tx)}{4},\\ \frac{G(fx,fy,Tx)}{2},\\ \frac{G(fx,Ty,Ty)+G(fy,Tx,Tx)+G(fz,Tx,Tx)}{5}\end{array}\right\} \qquad (5.40)$$

for all $x,y,z \in X$*, where* $k \in [0,\frac{1}{2})$*. Then* (f,T) *has a unique common fixed point.*

Corollary 5.2 *Let* (X,G) *be a G-metric space. Assume that* $f : X \to X$ *and* $T : X \to 2^X$ *be a pair of hybrid mappings. Then* (f,T) *is called a generalized Meir-keeler type contraction whenever for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\varepsilon \leq \psi(M(x,y,z)) < \varepsilon + \delta \implies H_G(Tx,Ty,Tz) < \varepsilon,$$

where

$$M(x,y,z) \;=\; \max\left\{\begin{array}{c}G(fx,fy,fz),G(Tx,fx,fx)\\ G(Ty,fy,fy),G(Tz,fz,fz)\end{array}\right\} \qquad (5.41)$$

for all $x,y,z \in X$*. Then* (f,T) *has a unique common fixed point.*

In the following section, we formulate a vivid example to validate Theorem 5.5.

Example 5.3 Let $X = [0,1]$ be equipped with G-metric, $G(x,y,z) = |x-y| + |y-z| + |x-z|$. Define $Tx = [0,\frac{x}{8(x+1)}]$, $fx = x^{\frac{3}{2}}$ and $x_n = \frac{1}{n}$. Also, $\psi = \frac{1}{2}t, t > 0$ for all $x,y,z \in X$.

Using Definition 5.6, the mappings f and T are called weakly compatible if $fx \in Tx$ for some $x \in X$ which implies that $fTx \subseteq Tfx$. To observe this consider, $Tx = [0,\frac{x}{8(x+1)}]$ and $fx = x^{\frac{3}{2}}$.

For

$$fTx \;=\; \left[\frac{x}{8(x+1)}\right]^{\frac{3}{2}},$$

and

$$Tfx = \left[0, \frac{x^{\frac{3}{2}}}{8(x^{\frac{3}{2}}+1)}\right],$$

so that

$$\left[\frac{x}{8(x+1)}\right]^{\frac{3}{2}} \subseteq \left[0, \frac{x^{\frac{3}{2}}}{8(x^{\frac{3}{2}}+1)}\right].$$

Also, f and T satisfy CLR_f property. By Definition 5.15 and let x_n be a sequence in X. Then

$$\begin{aligned} \lim_{n\to\infty} f\left(\frac{1}{n}\right) &= fu \in A = \lim_{n\to\infty} T\left(\frac{1}{n}\right), \\ \lim_{n\to\infty} f\left(\frac{1}{n}\right) &= f0 \in A = \lim_{n\to\infty} T\left(\frac{1}{n}\right). \end{aligned}$$

Next, we show that TX is a G-complete subspace in X. By Definition 2.16 we have

$$\begin{aligned} G(x,y,y) &\leq |x-y|+|y-y|+|x-y| \\ &= 2|x-y|, \end{aligned} \tag{5.42}$$

and

$$\begin{aligned} G(y,x,x) &\leq |y-x|+|x-x|+|y-x|, \\ &= 2|y-x|. \end{aligned} \tag{5.43}$$

Using (5.42) and (5.43) in (2.13), we get

$$\begin{aligned} d_G(x,y) &\leq 2|x-y|+2|y-x| \\ &= 4|x-y|. \end{aligned} \tag{5.44}$$

To prove (e), let $x,y,z \in X$. If $x=y=z=0$, then $Tx=Ty=Tz=0$ and $H_G(Tx,Ty,Tz)=0$. Our proof is done. If not, suppose that the value of x,y,z are not all zero.

For $x \leq y \leq z$, we have

$$H_G(Tx,Ty,Tz) = H_G\left(\left[0, \frac{x}{8(x+1)}\right], \left[0, \frac{y}{8(y+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right). \tag{5.45}$$

By (2.15)

$$\Longrightarrow \quad \max\left\{\begin{array}{l} \sup_{0\leq a\leq \frac{x}{8(x+1)}} G\left(a, \left[0, \frac{y}{8(y+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right), \\ \\ \sup_{0\leq b\leq \frac{y}{8(y+1)}} G\left(b, \left[0, \frac{x}{8(x+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right), \\ \\ \sup_{0\leq c\leq \frac{z}{8(z+1)}} G\left(a, \left[0, \frac{x}{8(x+1)}\right], \left[0, \frac{y}{8(y+1)}\right]\right) \end{array}\right\}.$$

Since $x \le y \le z$, then $\left[0, \frac{x}{8(x+1)}\right] \subseteq \left[0, \frac{y}{8(y+1)}\right] \subseteq \left[0, \frac{z}{8(z+1)}\right]$, using (6.40) implies that

$$\begin{aligned}
d_G\left(\left[0, \frac{x}{8(x+1)}\right], \left[0, \frac{y}{8(y+1)}\right]\right) &= 0, \\
d_G\left(\left[0, \frac{y}{8(y+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right) &= 0, \\
d_G\left(\left[0, \frac{x}{8(x+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right) &= 0.
\end{aligned}$$

Now for each $0 \le a \le \frac{x}{8(x+1)}$ and $d_G(x,y) = 4|x-y|$ in (2.16) we have

$$\begin{aligned}
G\left(a, \left[0, \frac{y}{8(y+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right) &= d_G\left(a, \left[0, \frac{y}{8(y+1)}\right]\right) + \\
&\quad d_G\left(\left[0, \frac{y}{8(y+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right) + \\
&\quad d_G\left(a, \left[0, \frac{z}{8(z+1)}\right]\right) = 0 \\
&\le 4a - \frac{4y}{8(y+1)} + 0 + 4a - \frac{4z}{8(z+1)}, \\
&= 8a - \frac{4y(z+1) + 4z(y+1)}{8(y+1)(z+1)}, \\
&\le \frac{8x}{8(x+1)} - \frac{4y(z+1) + 4z(y+1)}{8(y+1)(z+1)},
\end{aligned}$$

$$= \frac{8x(y+1)(z+1) - (4y(z+1) + 4z(y+1))(x+1)}{8(x+1)(y+1)(z+1)}. \tag{5.46}$$

Next, for each $0 \le b \le \frac{y}{8(y+1)}$ and $d_G(x,y) = 4|x-y|$ in (2.16) we have

$$\begin{aligned}
G\left(b, \left[0, \frac{x}{8(x+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right) &= d_G\left(b, \left[0, \frac{x}{8(x+1)}\right]\right) + \\
&\quad d_G\left(\left[0, \frac{x}{8(x+1)}\right], \left[0, \frac{z}{8(z+1)}\right]\right) + \\
&\quad d_G\left(b, \left[0, \frac{z}{8(z+1)}\right]\right), \\
&\le 4b - \frac{4x}{8(x+1)} + 0 + 4b - \frac{4z}{8(z+1)},
\end{aligned}$$

$$
\begin{aligned}
&= 8b-\frac{4x(z+1)+4z(x+1)}{8(x+1)(z+1)},\\
&\leq \frac{8y}{8(y+1)}-\frac{4x(z+1)+4z(x+1)}{8(x+1)(z+1)},\\
&= \frac{8y(x+1)(z+1)-(4x(z+1)+4z(x+1))(y+1)}{8(x+1)(y+1)(z+1)}
\end{aligned}
\tag{5.47}
$$

For $\sup_{0\leq a\leq \frac{y}{8(y+1)}}$, we have

$$
G\left(b,\left[0,\frac{x}{8(x+1)}\right],\left[0,\frac{z}{8(z+1)}\right]\right)
$$

$$
= \frac{8y(x+1)(z+1)-(4x(z+1)+4z(x+1))(y+1)}{8(x+1)(y+1)(z+1)}.
$$

Similarly, for each $0\leq c\leq \frac{z}{8(z+1)}$ and (2.16) gives

$$
\begin{aligned}
G\left(c,\left[0,\frac{x}{8(x+1)}\right],\left[0,\frac{y}{8(y+1)}\right]\right) &= d_G\left(c,\left[0,\frac{x}{8(x+1)}\right]\right)+\\
&\quad d_G\left(\left[\frac{x}{8(x+1)}\right],\left[0,\frac{y}{8(y+1)}\right]\right)+\\
&\quad d_G\left(c,\left[0,\frac{y}{8(y+1)}\right]\right),
\end{aligned}
$$

$$
\begin{aligned}
&\leq 4c-\frac{4x}{8(x+1)}+0+4c-\frac{4y}{8(y+1)},\\
&= 8c-\frac{4x(y+1)+4y(x+1)}{8(x+1)(y+1)},\\
&\leq \frac{8z}{8(z+1)}-\frac{4x(y+1)+4y(x+1)}{8(x+1)(y+1)}.\\
&= \frac{8z(x+1)(y+1)-(4x(y+1)+4y(x+1))(z+1)}{8(x+1)(y+1)(z+1)}.
\end{aligned}
\tag{5.48}
$$

As a result, we deduce

$$
\begin{aligned}
H_G(Tx,Ty,Tz) &\leq \max\left\{\begin{array}{l}
\sup_{0\leq a\leq \frac{x}{8(x+1)}} \frac{8x(y+1)(z+1)-(4y(z+1)+4z(y+1))(x+1)}{8(x+1)(y+1)(z+1)},\\
\sup_{0\leq b\leq \frac{y}{8(y+1)}} \frac{8y(x+1)(z+1)-(4x(z+1)+4z(x+1))(y+1)}{8(x+1)(y+1)(z+1)},\\
\sup_{0\leq c\leq \frac{z}{8(z+1)}} \frac{8z(x+1)(y+1)-(4x(y+1)+4y(x+1))(z+1)}{8(x+1)(y+1)(z+1)}
\end{array}\right\},\\
&= \frac{8z(x+1)(y+1)-(4x(y+1)+4y(x+1))(z+1)}{8(x+1)(y+1)(z+1)}.
\end{aligned}
\tag{5.49}
$$

On the other hand, we calculate the following G-metrics. By (2.17) and (2.18), we get

$$\begin{aligned}
G(fx,fy,fz) &= G\Big(x^{\frac{3}{2}},y^{\frac{3}{2}},z^{\frac{3}{2}}\Big),\\
&\leq \Big|x^{\frac{3}{2}}-y^{\frac{3}{2}}\Big|+\Big|y^{\frac{3}{2}}-z^{\frac{3}{2}}\Big|+\Big|x^{\frac{3}{2}}-z^{\frac{3}{2}}\Big|,\\
&= 2x^{\frac{3}{2}}-2z^{\frac{3}{2}}.\\
G(fx,Ty,fx) &= G\Big(x^{\frac{3}{2}},\Big[0,\frac{y}{8(y+1)}\Big],x^{\frac{3}{2}}\Big),\\
&\leq \inf\Big\{d_G\Big(x^{\frac{3}{2}},0\Big),d_G\Big(x^{\frac{3}{2}},\frac{y}{8(y+1)}\Big)\Big\},\\
&\leq \inf\Big\{4x^{\frac{3}{2}},4x^{\frac{3}{2}}-\frac{4y}{8(y+1)}\Big)\Big\},\\
&= 4x^{\frac{3}{2}}-\frac{y}{8(y+1)}.\\
G(fy,Tx,fy) &= G\Big(y^{\frac{3}{2}},\Big[0,\frac{4x}{8(x+1)}\Big],y^{\frac{3}{2}}\Big),\\
&\leq \inf\Big\{d_G\Big(y^{\frac{3}{2}},0\Big),d_G\Big(y^{\frac{3}{2}},\frac{x}{8(x+1)}\Big)\Big\},\\
&\leq \inf\Big\{4y^{\frac{3}{2}},4y^{\frac{3}{2}}-\frac{x}{8(x+1)}\Big)\Big\},\\
&= 4y^{\frac{3}{2}}-\frac{4x}{8(x+1)}.\\
G(fz,Ty,fz) &= G\Big(z^{\frac{3}{2}},\Big[0,\frac{y}{8(y+1)}\Big],z^{\frac{3}{2}}\Big),\\
&\leq \inf\Big\{d_G\Big(z^{\frac{3}{2}},0\Big),d_G\Big(z^{\frac{3}{2}},\frac{4y}{8(y+1)}\Big)\Big\},\\
&\leq \inf\Big\{4z^{\frac{3}{2}},4z^{\frac{3}{2}}-\frac{4y}{8(y+1)}\Big)\Big\},\\
&= 4z^{\frac{3}{2}}-\frac{4y}{8(y+1)}.\\
G(fy,Tz,fy) &= G\Big(y^{\frac{3}{2}},\Big[0,\frac{z}{8(z+1)}\Big],y^{\frac{3}{2}}\Big),\\
&\leq \inf\Big\{d_G\Big(y^{\frac{3}{2}},0\Big),d_G\Big(y^{\frac{3}{2}},\frac{z}{8(z+1)}\Big)\Big\},\\
&\leq \inf\Big\{4y^{\frac{3}{2}},4y^{\frac{3}{2}}-\frac{4z}{8(z+1)}\Big)\Big\},\\
&= 4y^{\frac{3}{2}}-\frac{4z}{8(z+1)}.\\
G(fx,Tz,fx) &= G\Big(x^{\frac{3}{2}},\Big[0,\frac{z}{8(z+1)}\Big],x^{\frac{3}{2}}\Big),\\
&\leq \inf\Big\{d_G\Big(x^{\frac{3}{2}},0\Big),d_G\Big(x^{\frac{3}{2}},\frac{z}{8(z+1)}\Big)\Big\},\\
&\leq \inf\Big\{4x^{\frac{3}{2}},4x^{\frac{3}{2}}-\frac{4z}{8(z+1)}\Big)\Big\},
\end{aligned}$$

$$= \quad 4x^{\frac{3}{2}} - \frac{4z}{8(z+1)}.$$

Applying Equation (5.24) we obtain

$$\begin{aligned} M(x,y,z) \quad &\leq \quad \max\left\{\begin{array}{c} 2x^{\frac{3}{2}} - 2z^{\frac{3}{2}}, 4x^{\frac{3}{2}} - \frac{4y}{8(y+1)}, 4y^{\frac{3}{2}} - \frac{4x}{8(x+1)}, 4z^{\frac{3}{2}} - \frac{4x}{8(x+1)}, \\ 4z^{\frac{3}{2}} - \frac{4y}{8(y+1)}, 4y^{\frac{3}{2}} - \frac{4z}{8(z+1)}, 4x^{\frac{3}{2}} - \frac{4z}{8(z+1)}, \end{array}\right\} \\ &= \quad 4z^{\frac{3}{2}} - \frac{4y}{8(y+1)}. \end{aligned} \tag{5.50}$$

By (5.50) and (5.49) it follows that

$$\frac{8z(x+1)(y+1) - (4x(y+1) + 4y(x+1))(z+1)}{8(x+1)(y+1)(z+1)} \quad \leq \quad \psi(4z^{\frac{3}{2}} - \frac{4y}{8(y+1)}).$$

This shows that all conditions imposed in Theorem 5.5 are satisfied. Hence a pair of hybrid mapping f and T in G-metric space has a unique common fixed point.

5.5 The common Fixed Point Results for p-Hybrid Mappings in G-Metric Spaces

Next, we present our second main result by extending Definition 5.12 using a pair of p-hybrid mapping in G-metric space concept.

Definition 5.16 Let (X,G) be a G-metric space and f,T be a pair of hybrid mapping on this space for $p \geq 0$ and $k_i \geq 0, i = 1,2,3,4$, such that $\sum_{i=1}^{4} k_i = 1$. We define the following expression.

$$\mathscr{M}_{\mathscr{G}}^{p}(\zeta,\eta,\eta) \quad = \begin{cases} \Big[k_1(G(f\zeta,f\eta,f\eta))^p + k_2(G(f\zeta,T\zeta,T\zeta))^p + k_3(G(T\eta,f\eta,f\eta))^p \\ \\ +k_4\Big(\frac{G(f\zeta,T\eta,T\eta)+G(T\zeta,f\eta,f\eta)}{2}\Big)^p\Big]^{\frac{1}{p}}, \text{ for } p \geq 0, \zeta,\eta \in X; \\ \\ \Big[G(f\zeta,f\zeta,f\eta)\Big]^{K_1}\Big[G(f\zeta,T\zeta,T\zeta)\Big]^{K_2}\Big[G(T\eta,f\eta,f\eta)\Big]^{K_3} \\ \Big[\frac{G(T\zeta,f\eta,f\eta)+G(f\zeta,T\eta,T\eta)}{2}\Big]^{K_4}, \text{ for } p = 0, \zeta,\eta \in X. \end{cases} \tag{5.51}$$

Theorem 5.6 Let (X,G) be a complete G- metric space and suppose mapping f,T : $X \to CB(X)$ is a p-hybrid contraction with almost altering distance $\psi \in \Psi$ satisfy the following conditions:

(a) f and T are weakly compatible;

(b) f and T satisfy CLR_f property;

(c) $TX \subseteq f(X)$;

(d) $T(X)$ is a G-complete subspace of X;

(e) $\mathscr{H}_G^p(T\zeta,T\eta,T\eta) \leq \psi(\mathscr{M}_{\mathscr{G}}^p(\zeta,\eta,\eta))$; for all $\zeta,\eta \in X$ and $p \geq 0$,

where

$$\begin{aligned}\mathscr{M}_{\mathscr{G}}^p(\zeta,\eta,\eta) &= \Big[k_1(G(f\zeta,f\eta,f\eta))^p + k_2(G(f\zeta,T\zeta,T\zeta))^p + k_3(G(T\eta,f\eta,f\eta))^p \\ &\quad + k_4\Big(\frac{G(f\zeta,T\eta,T\eta)+G(T\zeta,f\eta,f\eta)}{2}\Big)^p\Big]^{\frac{1}{p}}. \qquad (5.52)\end{aligned}$$

Then f and T admits a unique common fixed point in X.

Proof. Applying Definition 5.15, as the pair (f,T) satisfy CLR_f property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f\zeta_n = fw \in B = \lim_{n\to\infty} T\zeta_n,$$

for some $w \in X$ and $B \in CB(X)$.

We assume that $fw \in Tw$. If $fw \neq Tw$, for all $w \in X$, using (5.52) with $\zeta = \zeta_n$ and $\eta = w$, we have

$$H_G^p(T\zeta_n,Tw,Tw) \leq \psi(\mathscr{M}_{\mathscr{G}}^p(\zeta_n,w,w)), \qquad (5.53)$$

$$\begin{aligned}\mathscr{M}_{\mathscr{G}}^p(\zeta_n,w,w) &= \Big[k_1(G(f\zeta_n,fw,fw))^p + k_2(G(f\zeta_n,T\zeta_n,T\zeta_n))^p + k_3(G(Tw,fw,fw))^p \\ &\quad + k_4\Big(\frac{G(f\zeta_n,Tw,Tw)+G(T\zeta_n,fw,fw)}{2}\Big)^p\Big]^{\frac{1}{p}}. \qquad (5.54)\end{aligned}$$

Passing to the limit as $n \to \infty$ in (5.54) with $f\zeta_n = fw$, $B = T\zeta_n = Tw$, we obtain

$$\begin{aligned}\mathscr{M}_{\mathscr{G}}^p(\zeta_n,w,w) &\leq \Big[k_1(G(fw,fw,fw))^p + k_2(G(fw,Tw,Tw))^p + k_3(G(Tw,fw,fw))^p \\ &\quad + k_4\Big(\frac{G(fw,Tw,Tw)+G(Tw,fw,fw)}{2}\Big)^p\Big]^{\frac{1}{p}}, \\ &\leq \Big[(k_2+k_3+k_4)\Big(G(fw,Tw,Tw)\Big)^p\Big]^{\frac{1}{p}}, \\ &= (k_2+k_3+k_4)^{\frac{1}{p}} G(fw,Tw,Tw). \qquad (5.55)\end{aligned}$$

Using (5.55) in (5.53) as a result yields

$$H_G^p(B,Tu,Tu) \leq \psi((k_2+k_3+k_4)^{\frac{1}{p}} G(fw,Tw,Tw)). \qquad (5.56)$$

By the property of ψ, it follows that

$$H_G^p(B,Tw,Tw) < G(fw,Tw,Tw). \qquad (5.57)$$

Using (5.57) and Lemma 2.16, we get

$$\begin{aligned} G(fw,Tw,Tw) &\leq G(fw,Tw,Tw)+\varepsilon, \\ G(fw,Tw,Tw)-G(fw,Tw,Tw) &\leq \varepsilon, \\ \Rightarrow 0 &< \varepsilon, \end{aligned} \tag{5.58}$$

which is a contradiction. Hence, $fw \in Tw$ which shows that the pair (f,T) has a coincidence point.

Next, assume that $fw \in Tw$ and $fz \in Tz$. By Lemma 2.15 and Proposition 5.1, we have

$$G(fz,z,z) < G(Tz,Tz,Tz).$$

For the uniqueness of a point of coincidence of f and T. We follow a similar procedure as in the proof of Theorem 5.5. □

Theorem 5.7 *Let (X,G) be a complete G-metric space and suppose the mapping $f,T : X \to CB(X)$ is a 0-hybrid contraction with almost altering distance $\psi \in \Psi$ satisfy the following conditions:*

(a) *f and T satisfy CLR_f property;*

(b) *$\mathscr{H}_G^p(T\zeta,T\eta,T\eta) \leq \psi(\mathscr{M}_{\mathscr{G}}^p(\zeta,\eta,\eta))$; for all $\zeta,\eta \in X$ and $p = 0$ with $k_1+k_2+k_3+k_4 < 1$,*

where

$$\begin{aligned} \mathscr{M}_{\mathscr{G}}^p(\zeta,\eta,\eta) = {} & \Big[G(f\zeta,f\zeta,f\eta)\Big]^{K_1}\Big[G(f\zeta,T\zeta,T\zeta)\Big]^{K_2}\Big[G(T\eta,f\eta,f\eta)\Big]^{K_3} \\ & \Big[\frac{G(T\zeta,f\eta,f\eta)+G(f\zeta,T\eta,T\eta)}{2}\Big]^{K_4}, \end{aligned} \tag{5.59}$$

Then f and T admit a unique common fixed point in X.

Proof. By Definition 5.15, (f,T) satisfy CLR_f property, so there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} f\zeta_n = fw \in B = \lim_{n\to\infty} T\zeta_n,$$

for some $w \in X$ and $B \in CB(X)$.

Suppose that $fw \in Tw$. If $fw \neq Tw$, for all $w \in X$, using (5.51) with $\zeta = \zeta_n$ and $\eta = w$, we have

$$H_G^p(T\zeta_n,Tw,Tw) \leq \psi(M(\zeta_n,w,w)), \tag{5.60}$$

$$\begin{aligned}\mathscr{M}^{p}_{\mathscr{G}}(\zeta_n,w,w) &= \Big[G(f\zeta_n,f\zeta_n,fw)\Big]^{K_1}\Big[G(f\zeta_n,T\zeta_n,T\zeta_n)\Big]^{K_2}\Big[G(Tw,fw,fw)\Big]^{K_3}\\ &\quad \left[\frac{G(T\zeta_n,fw,fw)+G(f\zeta_n,Tw,Tw)}{2}\right]^{K_4}. \end{aligned} \tag{5.61}$$

Passing to the limit as $n \to \infty$ in (5.61) with $f\zeta_n = fw$, $B = T\zeta_n = Tw$, we obtain

$$\begin{aligned}\mathscr{M}^{p}_{\mathscr{G}}(\zeta_n,w,w) &\leq \Big[G(fw,fw,fw)\Big]^{K_1}\Big[G(fw,Tw,Tw)\Big]^{K_2}\Big[G(Tw,fw,fw)\Big]^{K_3}\\ &\quad \left[\frac{G(Tw,fw,fw)+G(fw,Tw,Tw)}{2}\right]^{K_4},\\ &= [G(fw,Tw,Tw)]^{(k_2+k_3+k_4)}. \end{aligned} \tag{5.62}$$

Using (5.62) in (5.60) as a result yields

$$H^{p}_{G}(B,Tu,Tu) \;\leq\; \psi([G(fw,Tw,Tw)]^{(k_2+k_3+k_4)}). \tag{5.63}$$

By the property of ψ, it follows that

$$H^{p}_{G}(B,Tw,Tw) \;<\; [G(fw,Tw,Tw)]^{(k_2+k_3+k_4)}. \tag{5.64}$$

Using (5.64) and Lemma 2.16, we get

$$\begin{aligned} G(fw,Tw,Tw) &\leq [G(fw,Tw,Tw)]^{(k_2+k_3+k_4)}+\varepsilon,\\ G(fw,Tw,Tw)-[G(fw,Tw,Tw)]^{(k_2+k_3+k_4)} &\leq \varepsilon,\\ \Longrightarrow 0 &< \varepsilon, \end{aligned} \tag{5.65}$$

which is a contradiction. Hence, $fw \in Tw$ which shows that the pair (f,T) has a coincidence point. From the proof of Theorem 5.6, we conclude that T is 0-hybrid contraction in G-metric space. Also, it satisfies all conditions of Theorem 5.7. □

5.6 An Application on Weak Partial b-Metric Space

In this section, we will discuss an approximation of a non-linear hybrid ordinary differential equation. Dhage (2006) named it as a hybrid differential equation with a linear perturbation of first type (HDE), which will validate Theorem 5.4 for two pairs of hybrid mapping in weak partial b-metric space.

First, we will define some essential notions which will help develop our results. One can see in (Pathak 2018, Burton 1998) and the reference therein.

Assume that $\mathscr{J} = [t_0, t_0 + a]$ of a real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$, $a > 0$ be given.

Consider in the function space $C(\mathscr{J},\mathbb{R})$ of continuous real valued functions defined on $\mathscr{J}$. Let us define a norm $\|.\|$ and order relation $\leq$ in $C(\mathscr{J},\mathbb{R})$ by

$$\|x\| = \sup_{t\in\mathscr{J}} |x(t)|,$$

$x \leq y \Leftrightarrow x(t) \leq y(t)$ for all $t \in \mathscr{J}$. Then, we see that $C(\mathscr{J},\mathbb{R})$ is a Banach space with respect to the partial order relation $\leq$.

The Hybrid differential equations have been investigated in different dimensions by several researchers, one can see Krasnoselski (1964), Burton (1998), Dhage (2000) and the references therein.

Consider the initial value problem (IVP) of first order ordinary non-linear differential equation (HDE).

$$\begin{cases} x^{'}(t) = f(t,x(t)) + g(t,x(t)), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{5.66}$$

for all $t \in \mathscr{J}$, where $f,g : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Also, Consider (IVP) of (HDE).

$$\begin{cases} x^{'}(t) + \lambda x(t) = \mu e^{-\lambda t} p(t,x(t)) + \tilde{f}(t,x(t)) + \tilde{g}(t,x(t)), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{5.67}$$

for all $t \in \mathscr{J}$, where $\tilde{f},\tilde{g} : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and

$$\begin{aligned} \tilde{f}(t,x) &= f(t,x) + \lambda x, \\ \tilde{g}(t,x) &= g(t,x) - \mu e^{-\lambda t} p(t,x), \end{aligned}$$

$\lambda \geq 0$ with $\mu \leq \frac{\lambda}{1-e^{-a}}$.

Pathak (2018) proved the following Lemma to satisfy HDE:

Lemma 5.1 *(Pathak 2018) A function $u \in C(\mathscr{J},\mathbb{R})$ is a solution of HDE (5.66) if and only if it is a solution of a non-linear integral equation*

$$x(t) = x_0\, e^{-\lambda(t-t_0)} + \mu e^{-\lambda t} \int_{t_0}^{t} p(s,x(s))ds + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s}[\tilde{f}(s,x(s)) + \tilde{g}(s,x(s))]ds. \tag{5.68}$$

for all $t \in \mathscr{J}$.

By Lemma 5.1, the *HDE* (5.68) is equivalent to the operator equation

$$x(t) \quad = \quad Px(t)+Qx(t). \tag{5.69}$$

for all $t \in \mathscr{J}$, where

$$Px(t) \quad = \quad x_0 e^{-\lambda(t-t_0)} + \mu e^{-\lambda t} \int_{t_0}^{t} p(s,x(s))ds, \tag{5.70}$$

$$Qx(t) \quad = \quad e^{-\lambda(t)} \int_{t_0}^{t} e^{\lambda s}[\tilde{f}(s,x(s))+\tilde{g}(s,x(s))]ds. \tag{5.71}$$

for all $t \in \mathscr{J}$.

Definition 5.17 *(Dhage 2014) An operator $T : E \to E$ is partially non-linear $\mathscr{D}$-contraction if there exists a $\mathscr{D}$-function ψ such that*

$$||Tx-Ty|| \le \psi(||x-y||),$$

for all comparable elements $x,y \in E$, where $0 < \psi(t) < t$ for $t > 0$.

From the continuity of integral, it follows that P and Q defines the maps $P,Q : E \to E$. The following applicable hybrid fixed point theorem is proved in Dhage (2000).

Theorem 5.8 *(Dhage 2000) Let $(E, \preceq, ||.||)$ be a regular partial ordered complete normed linear space such that the order relation $\preceq$ and the norm $||.||$ in E are compatible. Let $P,Q : E \to E$ be two nondencreasing operators such that*

(i) *P is partially bounded and partially non-linear $\mathfrak{D}$-contraction,*

(ii) *Q is partially Continuous and partially compact, and*

(iii) *there exists an element $x_0 \in E$ such that*

$$x(t) \quad \preceq \quad Px(t)+Qx(t).$$

Then the operator equation $x \preceq Px+Qx$ has a solution x^* in E and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by

$$x_{n+1} \quad = \quad Px_n+Qx_n, \ n = 0,1,2\ldots,$$

converge monotonically to x^*.

Consider the function space $C(\mathscr{J},\mathbb{R})$ of continuous real valued functions defined on $\mathscr{J}$. Let us define a norm $||.||$ of weak partial b-metric on M by

$$e_b(x,y) = \sup_{t\in\mathscr{J}} |x(t)-y(t)|^p + \alpha, \tag{5.72}$$

$\forall x,y \in C(\mathscr{J},\mathscr{R}), p > 1$ and $\alpha > 0$.

We rewrite the integral equation (5.68) in the form of a fixed point problem

$$x(t) = Tx(t).$$

For a map T defined by

$$Tx(t) \;=\; x_0(t) + \int_{t_0}^{t} K(s,x(s))ds,\; t \in [\mathscr{J},\mathbb{R}], \tag{5.73}$$

with

$$x_0(t) \;=\; x_0 e^{-\lambda(t-t_0)},$$

and

$$K(s,x(s)) \;=\; \mu e^{-\lambda t} p(s,x(s)) + e^{\lambda(s-t)}[\tilde{f}(s,x(s)) + \tilde{g}(s,x(s))].$$

Our main results of this section are as follows.

Theorem 5.9 *Let $(M,\preceq,\|.\|)$ be a weak partial b- ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|.\|$ in M are coincidentally idempotent. Let $f,g : M \to M$ and $G,T : M \to CB^{\rho_b}(M)$ be two hybrid pairs of non-decreasing operators such that*

(i) for any $x(t),y(t) \in C(\mathscr{J},\mathbb{R})$ there exists a $\mathscr{D}$-contraction function that satisfy

$$\|Gx(t) - Ty(t)\| \;\le\; (\psi(t))^p \|x(t) - y(t)\|^p + \alpha. \tag{5.74}$$

where $0 \le \psi(t) < 1$ and

$$\begin{aligned} (\psi(t))^p \|x(t) - y(t)\|^p + \alpha &= e^{-\tau} M(x,y), \\ M(x,y) &= F(\xi(N_1(x,y)) - \eta(N_1(x,y)) + LN_2(x,y)), \end{aligned}$$

where

$$\begin{aligned} N_1(x,y) \;=\; \max\Bigg\{ & \rho_b(gy,fx), \frac{\rho_b(fx,Gx) * \rho_b(gy,Ty)}{1+\rho_b(Gx,Ty)}, \\ & \frac{(\rho_b(gy,Gx))^2 + (\rho_b(fx,Ty))^2}{\rho_b(gy,Gx) + \rho_b(fx,Ty)}, \\ & \frac{\rho_b(fx,Gx) * \rho_b(fx,Ty) + \rho_b(gy,Ty) * \rho_b(gy,Gx)}{\rho_b(fx,Ty) + \rho_b(gy,Gx)} \Bigg\}, \end{aligned}$$

and

$$N_2(x,y) \;=\; \min\Big\{ \rho_b(fx,Gx), \rho_b(fx,Ty), \rho_b(gy,Gx), \rho_b(gy,Ty) \Big\},$$

Then Equation (5.68) has a fixed point $x \in M$.

Proof. Using equation (5.73) in (5.74) we obtain

$$\begin{aligned}
\|Gx(t)-Ty(t)\| &= \sup_{t\in\mathscr{J}}\left|\int_{t_0}^{t}[K(s,x(s))-K(s,y(s))]ds\right|^{p}+\alpha,\\
&\leq \sup_{t\in\mathscr{J}}\left[\Big(\int_{t_0}^{t}ds\Big)^{\frac{1}{q}}\left(\int_{t_0}^{t}|K(s,x(s))-K(s,y(s))|^{p}ds\right)^{\frac{1}{p}}\right]^{p}+\alpha,\\
&\leq \sup_{t\in\mathscr{J}}\Big(t-t_0\Big)^{\frac{p}{q}}\left(\int_{t_0}^{t}|K(s,x(s))-K(s,y(s))|^{p}ds\right)+\alpha,\\
&\leq \sup_{t\in\mathscr{J}}\Big(t-t_0\Big)^{p-1}\left(\int_{t_0}^{t}\psi(t)^{p}|x(t)-y(t)|^{p}ds\right)+\alpha,\\
&\leq \Big(t-t_0\Big)^{p-1}(t-t_0)\Big(\psi(t)^{p}|x(t)-y(t)|^{p}\Big)+\alpha,\\
&\leq \Big(t-t_0\Big)^{p}\Big(\psi(t)^{p}|x(t)-y(t)|^{p}\Big)+\alpha,\\
&\leq \Big(\Big(t-t_0\Big)\psi(t)\Big)^{p}|x(t)-y(t)|^{p}+\alpha,\\
&\leq (\psi(t))^{p}|x(t)-y(t)|^{p}+\alpha,\\
&\leq e^{-\tau}M(x,y).
\end{aligned}$$

For each $x,y\in X$, we have

$$\mathscr{H}^{+}_{\rho_b}(Gx,Ty) \leq e^{-\tau}\mathbb{M}(x,y). \tag{5.75}$$

Taking logarithms on both sides of (5.75) using $F_1(z)=\ln(z)$ and the property of F, we get

$$\ln(\mathscr{H}^{+}_{\rho_b}(Gx,Ty)) \leq \ln(e^{-\tau}\mathbb{M}(x,y)).$$

Equivalently to

$$\tau+F(\mathscr{H}^{+}_{\rho_b}(Gx,Ty)) \leq F(\mathbb{M}(x,y)).$$

Hence, the condition of hybrid differential equation (5.66) is satisfied and so Equation (5.68) has a solution. Therefore, the condition of Theorem (5.4) is validated for two pairs of hybrid mappings which are coincidentally idempotent. □

CHAPTER SIX

COINCIDENCE FIXED POINT THEOREMS FOR IMPLICIT CONTRACTION MAPPINGS

6.1 Introduction

Popa (1997) gave a generalisation of Banach contraction principle by introducing an implicit function whose strength lies in producing many contractions at once. It is still trending among the research community. For more details, we refer the readers to Imdad *et al.* (2002), Ali and Imdad (2008), Berinde and Vetro (2012), Imdad *et al.* (2016) and the references cited therein.

In similar way, Czerwik (1993) established b-metric spaces by weakening the triangle inequality coefficient and generalised Banach's contraction principle to these spaces. Since then, several papers are published in the fixed point theory of various classes of the single and multivalued map in b-metric space. One can see Kirk (2014), Roshan *et al.* (2016), Chifu and Petrusel (2017) and the references therein.

Likewise, Aamri and El-moutawakil (2002) introduced the (E.A.)-property for a pair of self-mappings defined on metric spaces. It contains the class of compatible and non-compatible mappings in metric spaces and utilised the same to prove common fixed point theorems under strict contractive conditions.

On the other hand, Karapinar (2013) initiated the concept of quasi-partial metric space and discussed the existence of fixed points of self-mapping on this space. Gupta and Gautam (2015, 2015b) further generalised the quasi-partial metric space to the class of quasi-partial b-metric spaces. Recently, Gautam and Verma (2021) discussed the fixed point results via implicit mapping in quasi-partial b-metric space.

Moreover, Ahmadullah *et al.* (2016c) proved a fixed point theorem for self mappings in metric-like spaces concerning binary relation. Eke *et al.* (2019) proved a common fixed point theorem for a pair of weakly compatible mappings under implicit contractive properties in metric spaces endowed with binary relation.

Finally, Nizar (2016) and Nizar and Nabil (2016) extended the concept of S-metric space to S_b-metric space and proved some fixed point results in partial S_b-metric spaces. Later, Mlaiki *et al.* (2017) proved the fixed point theorem for α-ψ-contractive mapping in S_b-metric spaces.

In general, this chapter deals with the results for coincidence point for a pair of self-mappings employing $(E.A.)$-property in metric-like spaces for implicit contractive mappings related to binary relation and common fixed point theorems for weakly compatible mappings satisfying an implicit relation in the setting of quasi-partial S_b-metric space, with some applications to differential and fractional differential equations.

Specifically, the chapter contains four sections; Section 6.2 gives the preliminary results of definitions, lemma and theorems, which will be used in developing the results. In Section 6.3, we prove the existence results for common fixed point theorems for a pair of self-mappings satisfying (E. A.)-property under binary relation via implicit contractive condition in metric -like spaces by extending Theorem 6.1 proved in Ahmadullah *et al.* (2016c) which contains parallel to the ideas based on Alam and Imdad (2015), Ahmadullah *et al.* (2016a) and Eke *et al.* (2019).

Next, Section 6.4 is motivated by the results of Gautam and Verma (2021) and Nizar (2016); we prove common fixed point theorems for weakly compatible mappings satisfying an implicit relation in the setting of quasi-partial S_b-metric space. Also, we obtain coincidence and a unique common fixed point of such mapping. Some examples are given to verify the validity of our results. Finally, in section 6.5, we give a discussion on some applications of Theorem 6.3 and 6.4 to differential and fractional differential equations.

6.2 Preliminaries

Now, we introduce some definitions related to a common fixed point in metric-like space with binary relations.

Definition 6.1 *(Jungck 1986) Let S,T be self-mappings of a non empty set X. A point $x \in X$ is coincidence points of S and T if $x^* = Sx = Tx$. The set of coincidence point of S and T is denoted by $C(S,T)$.*

Motivated from Jungck (1976) and Sessa (1982), we can have the following definitions:

Definition 6.2 *Let (S,T) be a pair of self mappings on a metric-like space (X,σ). Then a point $x^* \in X$ is called coincidence point of the pair (S,T) if $Tx_n = Sx_n = x^*$. If $x^* = x$ then, x is said to be a common fixed point.*

Definition 6.3 *Let (S,T) be a pair of self mappings on a metric-like space (X,σ). Then the pair (S,T) is said to be:*

(i) *Commuting if, for all $x \in X$, $S(Tx_n) = T(Sx_n)$,*

(ii) *Weakly commuting if, for all $\sigma(S(Tx_n), T(Sx_n)) \leq \sigma(Sx_n,Tx_n)$,*

(iii) *Compatible if $\lim\limits_{n\to\infty} \sigma(STx_n,TSx_n) = 0$, whenever x_n is a sequence in X such that $\lim\limits_{n\to\infty} Tx_n = \lim\limits_{n\to\infty} Sx_n = x^*$,*

(iv) *Weakly compatible if, for all $S(Tx_n) = T(Sx_n)$, for every coincidence point $x \in X$.*

Motivated from Aamri and Moutawakil (2002), we can have the following:

Definition 6.4 *A pair of self-mappings (T,S) of a metric-like space (X,σ) is said to satisfy the property (E.A) if there exist at least one sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x^*,$$

for some $x^* \in X$.

Remark 6.1 *It is known that two pairs of self-mappings S and T of a metric-like space (X,σ), will be non compatible if there exist a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x^*,$$

but

$$\lim_{n\to\infty} \sigma(STx_n,TSx_n)$$

is non zero or not exists.

Ahmadullah et al. (2016c) proved the results in metric-like spaces as well as partial metric spaces equipped with an arbitrary relation as follows:

Theorem 6.1 *(Ahmadullah* et al. *2016c) Let* $(X,\sigma,)$ *be a metric-like space equipped with a binary relation* $\mathscr{R}$ *defined on* X *and* f *a self-mapping on* X. *Suppose that the following conditions are satisfied:*

(a) *there exists a subset* $Y \subseteq X$ *with* $fX \subseteq Y$ *such that* (Y,σ) *is* $\mathscr{R}$*-complete,*

(b) *there exists* x_0 *such that* $(x_0, fx_0) \in \mathscr{R}$*,*

(c) $\mathscr{R}$ *is* f*-closed,*

(d) *either* f *is* $\mathscr{R}$*-continuous-like or* $\mathscr{R}|_Y$ *is* σ*-self-closed,*

(e) *there exists a constant* $k \in [0,1)$ *such that (* $\forall x,y \in X$ *with* $x,y \in \mathscr{R}$*)*

$$\sigma(fx,fy) \leq k\sigma(x,y). \tag{6.1}$$

Then f *has a fixed point. Moreover, if*

(f) $\Upsilon(fx,fy,\mathscr{R}^s)$ *is non-empty, for each* $x,y \in X$. *Then* f *has a unique fixed point.*

We introduce some definition related to a common fixed point via implicit mappings in quasi-partial metric space. For more detail we refer the reader to subsection 2.3.10.

Gautam and Verma (2021) proved the results for fixed point theorems of mappings satisfying implicit contractive relation in quasi-partial b-metric space. They considered the family F_Q be the set of all lower semi-continuous real functions. $F : \mathbb{R}^5_+ \to \mathbb{R}^+$ satisfying the following conditions:

$(F1)$ F is non-increasing in the t_1 and t_5 variable;

$(F2)$ for all $q,r \geq 0$, there exist $h \in [0,1)$ such that $F(q,r,r,q,s(q+r)) \leq 0$ implies $q \leq hr$;

$(F3)$ $F(t,t,0,0,t) > 0$ for all $t > 0$.

We give some examples of functions that satisfy the above implicit relation conditions.

Example 6.1 The function $F \in F_Q$ satisfies the properties (F1) - (F3) (see, (Gautam and Verma 2021)).

(1) $F(t_1,t_2,t_3,t_4,t_5) = t_1 - \alpha \max\{t_2,t_3,t_4,t_5\}$, where $\alpha \in [0, \frac{1}{2s})$;

(2) $F(t_1,t_2,t_3,t_4,t_5) = t_1 - a_1t_1 - a_2t_2 - a_3t_3 - a_4t_4 - a_5t_5$, where $a_i \geq 0$, i = 1,2,3,4, also $0 < a_1 + a_2 + a_3 + 2sa_4 < 1$ and $0 < a_1 + a_4 < 1$.

Gautam and Verma (2021) proved the following theorem satisfying implicit mappings.
Theorem 6.2 (Gautam and Verma 2021) Let (X, qp_b) be a complete quasi-partial b metric space and $T : X \to X$ is continuous self map for all $u \in X$. Suppose that

$$F\Big[qp_b(Tu,Tv), qp_b(u,v), qp_b(u,Tv), qp_b(v,Tv), [qp_b(u,Tv)+qp_b(v,Tu)]\Big] \leq 0.(6.2)$$

For some $F \in F_Q$ and if F satisfies $F(q,0,r,v,2sq) \leq 0$ for all $q,r \geq 0$, there exists $\beta \in [0, \frac{1}{s}]$ such that $u < \beta v$, then z is a unique fixed point of T. i.e, $Tz = z$ with $qp_b(z,z) = 0$.

Furthermore, Pathak *et al.* (2007), Abbas and Jungck (2008) gave the following definition for unique common fixed point notions.
Definition 6.5 *(Pathak et al. 2007, Abbas and Jungck 2008)*

(i) Let $\mathscr{S}$ and $\mathscr{A}$ be self maps of a set X. If $u^* = \mathscr{S}u = \mathscr{A}u$ for some u in X, then u is called a coincidence point of $\mathscr{S}$ and $\mathscr{A}$, and u^* is called a point of coincidence of $\mathscr{S}$ and $\mathscr{A}$.

(ii) Let $\mathscr{S}$ and $\mathscr{A}$ be weakly compatible self maps of a set X, we have $\mathscr{S}u^* = \mathscr{S}\mathscr{A}u = \mathscr{A}\mathscr{S}u = \mathscr{A}u^*$. If $\mathscr{S}$ and $\mathscr{A}$ have a unique point of coincidence $u^* = \mathscr{S}u = \mathscr{A}u$, then u^* is the unique common fixed point of $\mathscr{S}$ and $\mathscr{A}$.

6.3 Existence Results for Implicit Mappings under Binary Relation in Metric-like Spaces

In this section, we use some of definitions defined in Section 2.6 and Section 2.7 to prove the main results.

We prove the following theorem which is a generalisation and improvement of Theorem 6.1.
Theorem 6.3 Let (X, σ) be a metric-like space equipped with a binary relation $\mathscr{R}$ defined on X. Let T and S be a pair of self-mapping on X. Assume that the following conditions hold:

(a) there exists $TX \subseteq SX$ such that (X,σ) is $\mathscr{R}$- complete,

(b) there exists x_0 such that $(Sx_0,Tx_0) \in \mathscr{R}$,

(c) $X(T,S,\mathscr{R})$ is non-empty and satisfying (E. A.) property,

(d) either (T,S) is $\mathscr{R}$-continuous-like or $\mathscr{R}$ is σ-self-closed and weakly compatible,

(e) there exists an implicit function $F \in \mathscr{F}$ such that

$$F(\sigma(Tx,Ty),\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty),\sigma(Sx,Ty),\sigma(Sy,Tx)) \leq 0. \quad (6.3)$$

$\forall x,y \in X$ such that $x,y \in \mathscr{R}$. Then T and S has a common fixed point. Moreover, if

(f) $\Upsilon_{T,S}$ $(Tx,Sx,\mathscr{R}^s)$ is non-empty, for each $x,y \in X$, wherein F satisfies $(F3)$. Then T and S have a unique common fixed point.

Proof. Assume that $TX \subseteq SX$ and (X,σ) is $\mathscr{R}$-complete (definition 2.38), for x_0 with $(Sx_0,Tx_0) \in \mathscr{R}$. We can construct a T-S-sequence $\{Tx_n\}$ with initial point x_0 satisfying

$$(Sx_0,Tx_0),(Sx_1,Tx_1),(Sx_2,Tx_2),(Sx_3,Tx_3)\ldots(Sx_{2n},Tx_n),(Sx_{2n+1},Tx_{n+1}),$$

$\forall n \in \mathbb{N}_0$, such that, $\{Tx_{2n}\}$, $\{Sx_{2n}\} \in T(X)$.

From assumption (c), let x_0 be an arbitrary element of $X(T,S,\mathscr{R})$, then $(Sx_0,Tx_0) \in \mathscr{R}$. If $Sx_0 = Tx_0$, then x_0 is a common fixed point of T and S and proof is completed. Otherwise, if $Tx_0 \neq Sx_0$, then $SX \subset TX$. Now, we choose $x_1 \in X$ such that $Sx_1 = Tx_0$. Again, we can choose $x_2 \in X$ such that $Sx_2 = Tx_1$. Proceeding the same way, we construct a sequence $\{x_n\} \subset X$, such that

$$Sx_{2n+1} = Tx_{2n}, \forall n \in \mathbb{N}_0, \quad (6.4)$$

Now, we claim that $\{Tx_{2n}\}$ is $\mathscr{R}$-preserving sequence and using Definition 2.40, thus

$$(Tx_{2n},Tx_{2n+1}) \in \mathscr{R}, \forall n \in \mathbb{N}_0. \quad (6.5)$$

By mathematical induction, if $n = 0$ in (6.5) and using (c) such that $x_0 \in X(T,S,\mathscr{R})$, we have

$$(Sx_0,Sx_1) \in \mathscr{R}, \quad (6.6)$$

which proves that (6.5) is true for $n = 0$. Now, assume that (6.5) is true for $n = k > 0$ therefore

$$(Sx_{2k},Sx_{2k+1}) \in \mathscr{R}.$$

From condition (d), $\mathscr{R}$ is (T,S)-closed. From definition 2.41, we have

$$(Tx_{2k},Tx_{2k+1}) \in \mathscr{R},$$

which, on using (6.5) shows that

$$(Sx_{2k+1},Tx_{2k+2}) \in \mathscr{R},$$

therefore (6.6) holds for $n = 2k+1$. Hence, by induction, (6.6) is true for all $n \in \mathbb{N}$. In following (6.5) and (6.6), the sequence $\{Tx_n\}$ is also an $\mathscr{R}$-preserving, thus

$$(Tx_{2n},Tx_{2n+1}) \in \mathscr{R}, \forall n \in \mathbb{N}_0.$$

Also, assumption (c) claims that T and S satisfy $(E.A)$-property. Therefore, to prove the claim, let x_{2n} be a sequence in X, which is $\mathscr{R}$-preserving sequence. Using Definition (6.4), we have

$$\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n} = x^*. \tag{6.7}$$

$$\lim_{n\to\infty} \sigma(Tx_{2n},x) = \lim_{n\to\infty} \sigma(Sx_{2n},x). \tag{6.8}$$

By (σ_3) we have

$$\sigma(Tx_{2n},Sx_{2n}) \leqslant \sigma(Tx_{2n},x) + \sigma(Sx_{2n},x). \tag{6.9}$$

As $n \to \infty$, (6.9) leads to

$$\sigma(Tx_{2n},Sx_{2n}) \leq \sigma(Tx,x) + \sigma(Sx,x). \tag{6.10}$$

Which implies that

$$\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n} = x^* = x. \tag{6.11}$$

For some $x \in X$, suppose that SX is complete, then there exists $a \in X$ such that $x = Sa$. Using Equation (6.7), we have the following :

$$\lim_{n\to\infty} \sigma(Tx_{2n},Sa) = \lim_{n\to\infty} \sigma(Sx_{2n},Sa) = x^*. \tag{6.12}$$

Let us show that $Ta = Sa$. Suppose that $Ta \neq Sa$, using (6.3), we get

$$\begin{aligned} F(\sigma(Tx_{2n},Ta),\sigma(Sx_{2n},Sa),\sigma(Tx_{2n},Sx_{2n}),\sigma(Sa,Ta),& \\ \sigma(Sx_{2n},Ta),\sigma(Sa,Tx_{2n})) &\leq 0. \end{aligned} \tag{6.13}$$

Letting $n \to \infty$ in (6.13), we get

$$\begin{aligned} F(\sigma(Sa,Ta),\sigma(Sa,Sa),\sigma(Ta,Sa),\sigma(Sa,Ta),& \\ \sigma(Sa,Ta),\sigma(Sa,Ta)) &\leq 0. \end{aligned} \tag{6.14}$$

From (6.14), we have

$$\begin{aligned}\sigma(Ta,Sa) &\leq 0,\\ \Rightarrow \sigma(Ta,Sa) &= 0.\end{aligned}$$

Hence *Ta*=*Sa*, which is a contradiction.

Suppose that $TX \subset SX$. For every $x_0 \in X$ we consider the sequence $\{x_{2n}\} \in X$ defined by

$$\begin{aligned}Sx_{2n} &= Tx_{2n-1},\\ Sx_{2n+1} &= Tx_{2n}.\end{aligned}$$

Let $x_0 \in X$ be an arbitrary point. As $TX \subset SX$ one can choose T-S sequence $\{Tx_{2n}\}$ with initial point x_0. $x = x_{2n}$ and $y = x_{2n+1}$ in Equation (6.3) and denote

$$u = \sigma(Tx_{2n}, Tx_{2n+1}),$$

$$v = \sigma(Tx_{2n-1}, Tx_{2n}),$$

$$\begin{aligned}F(\sigma(Tx_{2n},Tx_{2n+1}),\sigma(Sx_{2n},Sx_{2n+1}),\sigma(Sx_{2n},Tx_{2n}),\sigma(Sx_{2n+1},Tx_{2n+1}),\\ \sigma(Sx_{2n},Tx_{2n+1}),\sigma(Sx_{2n+1},Tx_{2n})) \leq 0. \qquad (6.15)\end{aligned}$$

By substituting $Sx_n = Tx_{2n-1}$ and $Sx_{2n+1} = Tx_{2n}$ in Equation (6.15), we have

$$\begin{aligned}F(\sigma(Tx_{2n},Tx_{2n+1}),\sigma(Tx_{2n-1},Tx_{2n}),\sigma(Tx_{2n-1},Tx_{2n}),\sigma(Tx_{2n},Tx_{2n+1}),\\ \sigma(Tx_{2n-1},Tx_{2n+1}),\sigma(Tx_{2n},Tx_{2n})) \leq 0. \qquad (6.16)\end{aligned}$$

By substituting u, v in Equation (6.16), we obtain

$$F(u,v,v,u,\sigma(Tx_{2n-1},Tx_{2n+1}),0) \leq 0. \qquad (6.17)$$

Using (σ_3) and F_1, since is non-decreasing in the fifth variable, we get

$$\begin{aligned}\sigma(Tx_{2n-1},Tx_{2n+1}) &\leq \sigma(Tx_{2n-1},Tx_{2n}) + \sigma(Tx_{2n},Tx_{2n+1}),\\ \sigma(Tx_{2n-1},Tx_{2n+1}) &\leq v+u. \qquad (6.18)\end{aligned}$$

Using Equation (6.18) in (6.17), we have

$$F(u,v,v,u,u+v,0) \leq 0.$$

Which satisfies F_1, therefore

$$u \leq \lambda v.$$

Implies that

$$\begin{aligned}
\sigma(Tx_{2n},Tx_{2n+1}) &\leq \lambda\sigma(Tx_{2n-1},Tx_{2n}),\\
&\leq \lambda^{n}\sigma(Tx_0,Tx_1).\\
&\leq \lambda^{n+1}\sigma(x_0,x_1) \qquad (6.19)
\end{aligned}$$

Using (6.19) and (σ_3), for all $n,m \in \mathbb{N}_0$ with $m > n$, we obtain

$$\begin{aligned}
\sigma(Tx_{2n},Tx_{2m}) &\leq \sigma(Tx_{2n},Tx_{2n+1})+\sigma(Tx_{2n+1},Tx_{2n+2})+\ldots\sigma(Tx_{2m-1},Tx_{2m}),\\
&\leq (\lambda^{n}+\lambda^{n+1}+\lambda^{n+2}+\ldots+\lambda^{m-1})\sigma(Tx_0,Tx_1),\\
&\leq \lambda^{n}\sigma(Tx_0,Tx_1)(1+\lambda^{n}+\lambda^{n+1}+\lambda^{n+2}+\ldots+\lambda^{m-1}),\\
&\leq \frac{\lambda^{n}}{1-\lambda}\sigma(Tx_0,Tx_1),\\
&\rightarrow 0 \quad \text{as} \quad n\rightarrow\infty.
\end{aligned}$$

The results obey a Cauchy sequence properties of completeness. Hence $\{x_{2n}\}$ is $\mathscr{R}$-preserving Cauchy sequence. If the pair (T,S) is closed and weakly compatible, using Definition 6.3, we have

$$\begin{aligned}
TSz &= STz,\\
Tz &= Sz,\\
STz &= SSz,\\
TSz &= TTz.
\end{aligned}$$

To show that Tz is a common fixed point of T and S, we use inequality (6.3) which gives

$$\begin{aligned}
&F(\sigma(Tz,TTz),\sigma(Sz,STz),\sigma(Sz,Tz),\sigma(TTz,STz),\\
&\qquad\qquad \sigma(TTz,Sz),\sigma(Tz,STz)) \leq 0.
\end{aligned}$$

Which implies that

$$\begin{aligned}
&F(\sigma(Tz,TTz),\sigma(Tz,TTz),\sigma(Tz,Tz),\sigma(TTz,TTz),\\
&\qquad\qquad \sigma(TTz,Tz),\sigma(Tz,TTz)) \leq 0,\\
&\qquad\qquad\qquad\qquad \sigma(Tz,TTz) \leq 0. \qquad (6.20)
\end{aligned}$$

Thus $STz = TTz = Tz$. So Tz is a common fixed point of T and S.

For the uniqueness, take $z = Tz$ as a common fixed point of T and S. Assume that $w = Sw$ and $z \neq w$, using $x = z, y = w$ in Equation (6.3), we get

$$F(\sigma(Tz,Tw),\ \sigma(Sz,Sw),\ \sigma(Tz,Sz),\ \sigma(Tw,Sw),\ \sigma(Tw,Sz),\ \sigma(Tz,Sw)) \leq 0.$$

Hence, we get

$$\begin{aligned}
\sigma(z,w) &\leq 0,\\
\Rightarrow \sigma(z,w) &= 0,
\end{aligned}$$

which is a contradiction. Therefore, z is a unique common fixed point of T and S.

Using the assumption taken in Theorem 6.3, we prove assertion (f) as follows: we observe that $C(T,S)$ is non-empty, so let us take a pair of elements say (a,b) in $C(T,S)$ such that

$$\begin{aligned} Ta &= Sa = \bar{a}, \\ Tb &= Sb = \bar{b}. \end{aligned} \tag{6.21}$$

Next, we are required to show that $\bar{a} = \bar{b}$. By observing the above assertion, by definition 2.43, there exists a S-path (say, $z_0, z_1, z_2, ..., z_l$) of length l in $\mathscr{R}^s$ from Ta to Tb, with

$$\begin{aligned} Sz_0 &= Ta, \\ Sz_l &= Tb, \end{aligned} \tag{6.22}$$

such that

$$[Sz_{2i}, Sz_{2i+1}] \in \mathscr{R}^s \subseteq \mathscr{R}, \tag{6.23}$$

for all $i \in 0,1,2,3,...l-1$.

And

$$[Sz_{2i}, Tz_{2i}] \in \mathscr{R}^s \subseteq \mathscr{R} \tag{6.24}$$

for every $i \in 0,1,2,3,...l-1$.

Define two constant sequences such that

$$z_{2n}^0 = a \textit{ and } z_{2n}^l = b.$$

By using (6.22), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} Tz_{2n}^0 &= Ta = \bar{a}, \\ Tz_{2n}^l &= Tb = \bar{b}. \end{aligned}$$

By usual substitution for $z_0^i = z_i$ for each $i \in 0,1,2,...l$, that is

$$\begin{aligned} z_0^1 &= z_1, \\ z_0^2 &= z_2, \\ z_0^3 &= z_3, \\ z_0^4 &= z_4, \\ z_0^{l-1} &= z_{l-1}. \end{aligned}$$

Recall that $TX \subseteq SX$. Thus we construct a sequence

$$\{z_{2n}^1\}, \{z_{2n}^2\}, \{z_{2n}^3\}, ...\{z_{2n}^i\} \in X.$$

In general, $\{z_n^1\} \in X$

$$\begin{aligned}
Sz_{2n+1}^1 &= Tz_{2n}^1, \\
Sz_{2n+1}^2 &= Tz_{2n}^2, \\
Sz_{2n+1}^3 &= Tz_{2n}^3, \\
Sz_{2n+1}^4 &= Tz_{2n}^4, \\
Sz_{2n+1}^{l-1} &= Tz_{2n}^{l-1}, \forall n \in \mathbb{N}.
\end{aligned}$$

We obtain

$$Sz_{2n+1}^i = Tz_{2n}^i,$$

for all $i \in [0,\ l-1]$. Corresponding to each z_i, we have $[Sz_0^i, Sz_1^i] \in \mathscr{R}$ from (6.22), (6.23) and (T,S)-compactness of $\mathscr{R}$, we get

$$\lim_{n\to\infty} \sigma(Sz_{2n}^i, Sz_{2n+1}^i) = 0,$$

for each $i \in 1,2,3,...l-1$.

Thus, $\mathscr{R}$ is (T,S)-closed and we conclude that $[Tz_{2n}^i, Tz_{2n}^{i+1}] \in \mathscr{R}$, for each $i \in 0,1,2,3,...l-1$ and for all $n \in \mathbb{N}$.

Otherwise, $[Sz_{2n}^i, Sz_{2n}^{i+1}] \in \mathscr{R}$, for each $i \in 0,1,2,3,...l-1$ and for all $n \in \mathbb{N}$.

Define $\sigma_n^i = \sigma(Sz_{2n}^i, Sz_{2n}^{i+1})$, for each $i \in 0,1,2,3,...l-1$ and for all $n \in \mathbb{N}$. We assert that, $\lim_{n\to\infty} \sigma_{2n}^i > 0$. Assume that $\lim_{n\to\infty} \sigma_{2n}^i = \sigma > 0$.

Since $[Sz_{2n}^i, Sz_{2n}^{i+1}] \in \mathscr{R}$, either $[Sz_{2n}^i, Sz_n^{i+1}] \in \mathscr{R}$ or $[Sz_{2n}^{i+1}, Sz_{2n}^i] \in \mathscr{R}$.

If $[Sz_{2n}^i, Sz_{2n}^{i+1}] \in \mathscr{R}$, then applying the condition (e), we have

$$\begin{aligned}
F(\sigma(Tz_{2n}^i, Tz_{2n}^{i+1}), \sigma(Sz_{2n}^i, Sz_{2n}^{i+1}), \sigma(Sz_{2n}^i, Tz_{2n}^i), \sigma(Sz_{2n}^{i+1}, Tz_{2n}^{i+1}), \\
\sigma(Sz_{2n}^i, Tz_{2n}^{i+1}), \sigma(Sz_{2n}^{i+1}, Sz_{2n}^i)) \leq 0.
\end{aligned}$$

or

$$\begin{aligned}
F(\sigma(Sz_{2n+1}^i, Tz_{2n+1}^{i+1}), \sigma(Sz_{2n}^i, Sz_{2n}^{i+1}), \sigma(Sz_{2n}^i, Tz_{2n+1}^i), \sigma(Sz_{2n}^{i+1}, Sz_{2n+1}^{i+1}), \\
\sigma(Sz_{2n}^i, Sz_{2n+1}^{i+1}), \sigma(Sz_{2n}^{i+1}, Sz_{2n}^i)) \leq 0. \qquad (6.25)
\end{aligned}$$

Taking lim as $n \to \infty$ and using $\lim_{n\to\infty} \sigma_{2n}^i = \sigma$, we get

$$F(\sigma, \sigma, 0, 0, \sigma, \sigma) \leq 0.$$

Which is contradiction and hence

$$\lim_{n\to\infty} \sigma_{2n}^i = \sigma = 0.$$

The same, if $(Sz_{2n}^{i+1}, Sz_{2n+1}^{i}) \in \mathscr{R}$, we have

$$\lim_{n\to\infty} \sigma_{2n}^{i} = \lim_{n\to\infty} \sigma(Sz_{2n}^{i+1}, Sz_{2n+1}^{i}) = 0,$$

for $i \in 0,1,2,...l-1$.

Using (6.23), $\lim_{n\to\infty} \sigma_{2n}^{i} = 0$ and (σ_3), we have

$$\begin{aligned} \sigma(\bar{a},\bar{b}) = \sigma(Sz_{2n}^{0}, Sz_{2n}^{i}) &\leq \sum_{i=0}^{l-1} \sigma(Sz_{2n}^{i}, Sz_{2n}^{i+1}) \\ &\leq \sum_{i=0}^{l-1} \sigma_{2n}^{i}, \\ &\to 0 \;\; as\, n \to \infty. \end{aligned}$$

So that

$$\begin{aligned} \sigma(\bar{a},\bar{b}) &= 0 \\ \Longrightarrow \bar{a} &= \bar{b}. \end{aligned}$$

Therefore,

$$Sx = Sy.$$

Next we show the existence of common fixed point of T and S. Let $a \in C(T,S)$, i.e., by Definition 2.22, $Ta = Sa$. Proceeding using Definition 6.3, we have

(i) commuting if, for all $a \in X$,

$$\begin{aligned} S(Ta) &= T(Sa), \\ Sa &= Ta. \end{aligned}$$

(ii) Weakly commuting if, for all $d(S(Tx), T(Sx)) \leq d(Sx, Tx)$,

$$\sigma(S(Ta), T(Sa)) \leq \sigma(Sa, Ta).$$

(iii) Compatible if

$$\begin{aligned} \lim_{n\to\infty} d(STx_{2n}, TSx_{2n}) &= 0, \\ \sigma(S(Ta), T(Sa)) &= 0, \\ \sigma(Sa, Ta) &= 0, \\ Sa &= Ta. \end{aligned}$$

Also,

$$\begin{aligned} &\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n} = x^*, \\ &x_{2n} = a, Ta = Sa = x^* = a. \end{aligned}$$

(iv) Weakly compatible if, for all $S(Tx) = T(Sx)$, for every coincidence point $x \in X$.

$$S(Ta) = T(Sa).$$

If we take another point say b, let $Ta = b = Sa$, we obtain

$$\begin{aligned} S(Ta) &= T(Sa). \\ Sb &= Tb. \end{aligned}$$

So that, b is a common fixed point of T and S.

For uniqueness of common fixed point of T and S,

$$b = Sb = Sz = z.$$

Thus z is a common fixed point of T and S, we have $z = b$. Thus the proof completed.

□

From Theorem 6.3, we can deduce several corollaries which are itemised below:

Corollary 6.1 Let (X,σ) be a complete metric-like space and $T,S : X \to X$ be a pair of self mappings. Suppose that all the conditions of Theorem 6.3 holds, for all $(x,y \in X$ with $(Tx,Sy) \in \mathsf{R})$, we can obtain several results if the implicit relation (e) is replaced by one of the following:

(i)

$$\sigma(Tx,Ty) \leq k\sigma(Sx,Sy), \tag{6.26}$$

where $k \in [0,1)$.

(ii)

$$\sigma(Tx,Ty) \leq k[\sigma(Sx,Tx)+\sigma(Sy,Ty)], \tag{6.27}$$

where $k \in [0,\frac{1}{2})$.

(iii)

$$\sigma(Tx,Ty) \leq k[\sigma(Sx,Ty)+\sigma(Sy,Tx)], \tag{6.28}$$

where $k \in [0,\frac{1}{2})$.

(iv)

$$\begin{aligned}\sigma(Tx,Ty) &\leq a_1\sigma(Sx,Sy)+a_2[\sigma(Sx,Tx)+\sigma(Sy,Ty)]+\\ &\qquad a_3[\sigma(Sx,Ty)+\sigma(Sy,Tx)], \qquad (6.29)\end{aligned}$$

where $a_1,a_2,a_3 \in [0,1)$ and $a_1+2a_2+2a_3<1$.

(v)

$$\begin{aligned}\sigma(Tx,Ty) &\leq kd(Sx,Sy)+L\min\{\sigma(Sx,Tx),\sigma(Sy,Ty),\\ &\qquad \sigma(Sx,Ty),\sigma(Sy,Tx)\}, \qquad (6.30)\end{aligned}$$

where $k \in [0,1)$ and $L \geq 0$.

(vi)

$$\begin{aligned}\sigma(Tx,Ty) &\leq (a_1\sigma(Sx,Sy)+a_2\sigma(Sx,Tx)+a_3\sigma(Sy,Ty)+\\ &\qquad a_4(\sigma(Sx,Ty)+\sigma(Sy,Tx))) \qquad (6.31)\end{aligned}$$

where $a_1,a_2,a_3,a_4 \geq 0$ and $a_1+a_2+a_3+2a_4<1$.

(vii)

$$\begin{aligned}\sigma(Tx,Ty) &\leq k\max\Big\{\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty),\\ &\qquad \frac{\sigma(Sx,Tx)+\sigma(Sy,Ty)}{2}\Big\}+L\min\Big\{\sigma(Sx,Tx),\sigma(Sy,Ty),\\ &\qquad \sigma(Sx,Ty),\sigma(Sy,Tx)\Big\}, \qquad (6.32)\end{aligned}$$

where $k \in [0,1)$ and $L \geq 0$.

(viii)

$$\begin{aligned}\sigma(Tx,Ty) &\leq k\max\{\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty),\\ &\qquad \sigma(Sx,Ty),\sigma(Sy,Tx)\}, \qquad (6.33)\end{aligned}$$

where $k \in [0,\frac{1}{2})$.

(ix)

$$\begin{aligned}\sigma(Tx,Ty) &\leq k\max\big\{a_1\sigma(Sx,Sy)+a_2\sigma(Sx,Tx)+a_3\sigma(Sy,Ty)+\\ &\qquad a_4\sigma(Sx,Ty)+a_5\sigma(Sy,Tx)\big\},\end{aligned} \tag{6.34}$$

where $a_i^{\prime s}, \geq 0$ (for $i=1,2,3,4,5$) and $\sum_{i=1}^{5} a_i < 1$.

(x)

$$\begin{aligned}\sigma(Tx,Ty) &\leq k\max\big\{\sigma(Sx,Sy)+\sigma(Sx,Tx)+\sigma(Sy,Ty)+\\ &\qquad \frac{\sigma(Sx,Ty)}{2}+\frac{\sigma(Sy,Tx)}{2}\big\},\end{aligned} \tag{6.35}$$

where $k \in [0,1)$.

(xi)

$$\begin{aligned}\sigma(Tx,Ty) &\leq k\max\big\{\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty)\big\}+\\ &\qquad (1-k)\big\{a\sigma(Sx,Ty)+b\sigma(Sy,Tx)\big\},\end{aligned} \tag{6.36}$$

where $k \in [0,1)$ and $a,b < \frac{1}{2}$.

(xii)

$$\begin{aligned}\sigma(Tx,Ty)^2 &\leq \sigma(Tx,Ty)\big\{a_1\sigma(Sx,Sy),a_2\sigma(Sx,Tx),a_3\sigma(Sy,Ty)\big\}+\\ &\qquad a_4\sigma(Sx,Ty)\sigma(Sy,Tx),\end{aligned} \tag{6.37}$$

where $a_1 > 0$; $a_2,a_3,a_4 \geq 0$; $a_1+a_2+a_3 < 1$ and $a_1+a_4 < 1$.

(xiii)

$$\begin{aligned}\sigma(Tx,Ty)^2 &\leq a_1\max\big\{\sigma(Sx,Sy)^2,\sigma(Sx,Tx)^2,\sigma(Sy,Ty)^2\big\}+\\ &\qquad a_2\max\Big\{\sigma(Sx,Tx)\sigma(Sx,Ty),\sigma(Sy,Ty)\sigma(Sy,Tx)\Big\}\\ &\qquad -a_3\sigma(Sx,Ty)\sigma(Sy,Tx),\end{aligned} \tag{6.38}$$

where $a_i^{\prime s}, \geq 0$ (for $i=1,2,3$); $a_1+2a_2 < 1$ and $a_1+a_4 < 1$.

(xiv)

$$\begin{aligned}\sigma(Tx,Ty)^3 &\leq k\{\sigma(Sx,Sy)^3+\sigma(Sx,Tx)^3+\sigma(Sy,Ty)^3+\\ &\quad \sigma(Sx,Ty)^3+\sigma(Sy,Tx)^3\},\end{aligned} \tag{6.39}$$

where $k\in[0,\frac{1}{11})$.

Example 6.2 Consider $X=[0,2]$ endowed with complete metric-like, defined by metric $\sigma(x,y)=(x-y)^2$ in $\mathbb{R}^2$ with binary relation

$$\mathscr{R}=\{(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)\}\ on\ X.$$

Then X is either complete or $\mathscr{R}$-complete.

Define a pair of mappings $T,S:X\longrightarrow X$ by

$$Tx\ =\ \frac{x}{2},\forall x\in X,$$

and

$$Sx\ =\ x^2,\forall x\in X.$$

Then $TX=\{0\}\subset[0,\frac{1}{2}]\subseteq[0,2)=SX$.

Clearly, $\mathscr{R}$ is (T,S)-closed, and $x_0=0,(S0,T0)\in\mathscr{R}$.

Define continuous function $F:\mathbb{R}^6_+\to\mathbb{R}$ by

$$F(u_1,u_2,u_3,u_4,u_5,u_6)\ =\ u_1-\frac{1}{2}u_5-\frac{1}{2}u_6.$$

i.e.,

$$\sigma(Tx,Ty)\ \leq\ \frac{1}{2}\sigma(Sx,Ty)+\frac{1}{2}\sigma(Sy,Tx).$$

For

$$(x,y)\ \in\ \{(0,0)(0,1),(0,2),(1,1),(1,2),(2,2)\},\forall x,y\in\mathscr{R}.$$

$$\sigma(Tx,Ty)\ =\ 0,$$

hence obvious.

For $(x,y) \in (0,1)$

$$\begin{aligned}
\sigma(Tx,Ty) &= \sigma(T0,T1) = \frac{1}{4}.\\
\sigma(Sx,Ty) &= \sigma(S0,T1) = \frac{1}{4}.\\
\sigma(Sy,Tx) &= \sigma(S1,T0) = 1.
\end{aligned}$$

$$\begin{aligned}
\sigma(T0,T1) &\leq \frac{1}{2}\sigma(S0,T1) + \frac{1}{2}\sigma(S1,T0).\\
\frac{1}{4} &\leq \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times 1.\\
\frac{1}{4} &\leq \frac{1}{8} + \frac{1}{2}.\\
\frac{1}{4} &\leq \frac{5}{8}.
\end{aligned}$$

For $(x,y) \in (0,2)$

$$\begin{aligned}
\sigma(Tx,Ty) &= \sigma(T0,T2) = 1.\\
\sigma(Sx,Ty) &= \sigma(S0,T2) = 1.\\
\sigma(Sy,Tx) &= \sigma(S2,T0) = 16.
\end{aligned}$$

$$\begin{aligned}
\sigma(T0,T2) &\leq \frac{1}{2}\sigma(S0,T2) + \frac{1}{2}\sigma(S2,T0)\\
1 &\leq \frac{1}{2} \times 1 + \frac{1}{2} \times 16\\
1 &\leq \frac{1}{2} + 8\\
1 &\leq \frac{17}{2}.
\end{aligned}$$

For $(x,y) \in (1,1)$

$$\begin{aligned}
\sigma(Tx,Ty) &= \sigma(T1,T1) = 0.\\
\sigma(Sx,Ty) &= \sigma(S1,T1) = \frac{1}{4}.\\
\sigma(Sy,Tx) &= \sigma(S1,T1) = \frac{1}{4}.
\end{aligned}$$

$$\begin{aligned}
\sigma(T1,T1) &\leq \frac{1}{2}\sigma(S1,T1) + \frac{1}{2}\sigma(S1,T1)\\
0 &\leq \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4}\\
1 &\leq \frac{1}{8} + \frac{1}{8}\\
0 &\leq \frac{1}{4}.
\end{aligned}$$

For $(x,y) \in (1,2)$

$$\begin{aligned}
\sigma(Tx,Ty) &= \sigma(T1,T2) = \frac{1}{4}.\\
\sigma(Sx,Ty) &= \sigma(S1,T2) = 0.\\
\sigma(Sy,Tx) &= \sigma(S2,T1) = \frac{49}{4}.
\end{aligned}$$

$$\begin{aligned}
\sigma(T1,T2) &\leq \frac{1}{2}\sigma(S1,T2) + \frac{1}{2}\sigma(S2,T1)\\
\frac{1}{4} &\leq \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{49}{4}\\
\frac{1}{4} &\leq 0 + \frac{49}{8}\\
\frac{1}{4} &\leq \frac{49}{8}.
\end{aligned}$$

For $(x,y) \in (2,2)$

$$\begin{aligned}
\sigma(Tx,Ty) &= \sigma(T2,T2) = 0.\\
\sigma(Sx,Ty) &= \sigma(S2,T2) = 9.\\
\sigma(Sy,Tx) &= \sigma(S2,T2) = 9.
\end{aligned}$$

$$\begin{aligned}
\sigma(T2,T2) &\leq \frac{1}{2}\sigma(S2,T2) + \frac{1}{2}\sigma(S2,T2)\\
0 &\leq \frac{1}{2} \times 9 + \frac{1}{2} \times 9\\
0 &\leq \frac{9}{2} + \frac{9}{2}\\
0 &\leq 9.
\end{aligned}$$

This shows that all assertions of Theorem 6.3 are satisfied. Hence $x = 0$ is a fixed point of T.

Furthermore, using Equation 6.3, we deduce an implicit function as shown below

$$\begin{aligned}
&F(\sigma(Tx,Ty), \sigma(Sx,Sy), \sigma(Sx,Tx), \sigma(Sy,Ty), \sigma(Sx,Ty), \sigma(Sy,Tx))\\
&\qquad = \sigma(Tx,Ty) - \frac{1}{2}[\sigma(Sx,Ty) + \sigma(Sy,Tx)]
\end{aligned}$$

$$
\begin{aligned}
&= \sigma(Tx,Ty)-\frac{1}{2}[\sigma(Sx,Ty)+\sigma(Sy,Tx)]\\
&= (\frac{x}{2},\frac{y}{2})-\frac{1}{2}[\sigma(x^2,\frac{y}{2})+\sigma(y^2,\frac{x}{2})],\\
&= (\frac{x}{2}-\frac{y}{2})^2-\frac{1}{2}[(x^2-\frac{y}{2})^2+(y^2-\frac{x}{2})^2],\\
&= \frac{1}{4}x^2-\frac{xy}{2}+\frac{y^2}{4}-\frac{1}{2}x^4+\frac{x^2y}{2}-\frac{y^2}{8}-\frac{y^4}{2}+\frac{xy^2}{2}-\frac{x^2}{8},\\
&= \frac{1}{8}x^2+\frac{1}{8}y^2+\frac{x^2y}{2}+\frac{xy^2}{2}-\frac{xy}{2}-\frac{x^4}{2}-\frac{y^4}{2},\\
&= \frac{1}{8}[x^2+y^2+4x^2y+4xy^2-4xy-4x^4-4y^4].
\end{aligned}
$$

which is the implicit function satisfying Theorem 6.3.

6.4 Existence Results for Implicit Mappings in Quasi-partial-S_b Metric Spaces

In this section, we prove the existence results for implicit mappings in quasi-partial-S_b metric spaces. First, we recall some definitions on quasi-partial-S_b metric space from section 2.3.11 which we will use in developing our main results.

Inspired by Nizar (2016), Nizar and Nabil (2016), and Gautam and Verma (2021), we introduce the concept of quasi partial S_b-metric space as follows:

Definition 6.6 *A quasi-partial S_b-metric on a non empty set X is a mapping $Sqp_b : X\times X\times X\to\mathbb{R}^+$ such that for some real number $s\geq 1$ and all $u,v,z\in X$:*

$(QPSb1)$ $Sqp_b(u,u,u)=Sqp_b(u,v,z)=Sqp_b(v,v,y)\Rightarrow u=v=z;$

$(QPSb2)$ $Sqp_b(u,u,v)=Sqp_b(v,v,u);$

$(QPSb2)$ $Sqp_b(u,u,u)\leq Sqp_b(u,u,v)$*; and*

$(QPSb4)$ $Sqp_b(u,v,z)\leq s[Sqp_b(u,u,t)+Sqp_b(v,v,t)+Sqp_b(z,z,t)]-Sqp_b(t,t,t).$

A quasi-partial S_b-metric space is a pair (X,Sqp_b) such that X is a non-empty set and (X,Sqp_b) is a quasi partial S_b-metric on X. The number s is called the coefficient of (X,Sqp_b).

For a quasi-partial S_b-metric space (X,Sqp_b), the function $d_{Sqpb}:X\times X\times X\to R^+$ defined by $d_{Sqp_b}(u,u,v)=Sqp_b(u,u,v)+Sqp_b(v,v,u)-Sqp_b(u,u,u)-Sqp_b(v,v,v,)$ is a S_{qp_b}-metric on X.

The following are fundamental convergence properties of quasi- partial S_b- metric spaces.

Definition 6.7 *Let* (X, S_{qp_b}) *be a quasi-partial* S_b *metric space, then:*

(i) a sequence $\{u_n\} \subset X$ *converges to a point* $u \in X$ *if and only if*

$$Sqp_b(u,u,u) \;=\; \lim_{n\to\infty} Sqp_b(u_n,u_n,u) = \lim_{n\to\infty} Sqp_b(u,u,u_n),$$

(ii) a sequence $\{u_n\}$ *of elements of X is called a Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} Sqp_b(u_n,u_n,u_m) \text{ and } \lim_{n,m\to\infty} Sqp_b(u_m,u_m,u_n)$$

exist and are finite,

(iii) the quasi- partial S_b*-metric space* (X, Sqp_b) *is said to be complete if every Cauchy sequence* $\{u_n\} \subset X$ *converges to a point* $u \in X$ *such that*

$$\lim_{n,m\to\infty} Sqp_b(u_n,u_n,u_m) = \lim_{n,m\to\infty} Sqp_b(u_m,u_m,u_n) = Sqp_b(u,u,u).$$

Lemma 6.1 *Let* (X, Sqp_b) *be a quasi-partial b-metric space. Then the following hold:*

(i) If $Sqp_b(u,u,u) = 0$*, then* $u = v$*.*

(ii) If $u \neq v$*, then* $Sqp_b(u,u,v) > 0$ *and* $Sqp_b(v,v,u) > 0$*.*

From, Sedghi *et al.* (2012), we prove the following lemma to satisfy quasi -partial S_b-metric space.

Lemma 6.2 In a S_{qpb}-metric space, we have

$$S_{qpb}(u,u,v) = S_{qpb}(v,v,u).$$

Proof. By condition $(QPSb4)$ of Definition 6.6 and $u = t$ we get

$$\begin{aligned} S_{qpb}(u,u,v) &\le s[S_{qpb}(u,u,t) + S_{qpb}(u,u,t) + S_{qpb}(v,v,t)] - S_{qpb}(t,t,t) \\ &\le s[0+0+S_{qpb}(v,v,t)] - 0 \\ &= sS_{qpb}(v,v,t). \end{aligned} \tag{6.40}$$

Similarly,

$$\begin{aligned} S_{qpb}(v,v,u) &\le s[S_{qpb}(v,v,t) + S_{qpb}(v,v,t) + S_{qpb}(u,u,t)] - S_{qpb}(t,t,t) \\ &\le s[0+0+S_{qpb}(xu,u,t)] - 0 \\ &= sS_{qpb}(u,u,t). \end{aligned} \tag{6.41}$$

Consequently, by (6.40) and (6.41) as a result

$$S_{qpb}(u,u,t) \;=\; S_{qpb}(v,v,t).$$

□

Example 6.3 Let $X = [0,1]$. Define $S_{qp_b} : X \times X \times X \times \to \mathbb{R}^+$ as $S_{qp_b}(u,v,z) = (u-v)^2 + (v-z)^2 + u + v$. It is easy to show that (X, S_{qp_b}) is a quasi - partial S_b metric space.

By $(QPSb1)$, for $u = v = z$ we have $S_{qpb}(u,u,u) = S_{qpb}(v,v,v) = S_{qpb}(z,z,z)$

$$\begin{aligned} S_{qpb}(u,u,u) &\leq (u-v)^2 + (v-z)^2 + u + v, \\ &= (u-u)^2 + (u-u)^2 + u + u, \\ &= 2u. \end{aligned}$$

By $(QPSb2)$, for all $u, v \in X$ we have

$$\begin{aligned} S_{qpb}(u,u,v) &\leq (u-u)^2 + (u-v)^2 + u + u, \\ &= (u-v)^2 + u + u, \\ &= u^2 - 2uv + v^2 + 2u, \end{aligned}$$

and

$$\begin{aligned} S_{qpb}(v,v,u) &\leq (v-v)^2 + (v-u)^2 + v + v, \\ &= (v-u)^2 + v + v, \\ &= v^2 - 2uv + u^2 + 2v, \end{aligned}$$

hence, $v^2 - 2uv + v^2 + 2u = v^2 - 2uv + u^2 + 2v$.

Similarly, $(QPSb3)$ follows from $(QPSb2)$ and $(QPSb1)$

$$2u \leq u^2 - 2uv + v^2 + 2u.$$

Cosenquently, by $(QPSb4)$, we get

$$\begin{aligned} S_{qpb}(u,u,t) &= (u-t)^2 + 2u, \\ S_{qpb}(v,v,t) &= (v-t)^2 + 2v, \\ S_{qpb}(z,z,t) &= (z-t)^2 + 2z. \end{aligned}$$

Combining all the above inequalities using $(QPSb4)$, we obtain

$$(u-v)^2 + (v-z)^2 + u + v \leq s[(u-t)^2 + 2u + (v-t)^2 + 2v] - ((z-t)^2 + 2z),$$

thus, all axioms are satisfied. Hence (X, S_{qp_b}) is complete.

We are Motivated by the concept given by Gautam and Verma 2021 above. We introduce a definition related to a common fixed point via implicit mappings in quasi-partial S_b-metric space as follows.

Definition 6.8 *Consider* $s \geq 1$. *Let* F_Q *be the set of all functions* $F_S(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}^5 \to \mathbb{R}$ *such that*

$(FS1)$ *F_S is non-increasing in the t_1 and t_5 variable;*

$(FS2)$ *for all $q, r \geq 0$, there exist $\vartheta \in [0, \frac{1}{s}]$, such that $F_S(q, r, q, r, s(2q+r)) \leq 0$ implies $q \leq \vartheta r$;*

$(FS3)$ *$F_S(t, t, 0, 0, t) > 0$ for all $t > 0$.*

Example 6.4 The functions $F_S \in F_Q$ satisfy the properties (FS1) - (FS3).

(i) $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - t_5$, where $\gamma \in [0, \frac{1}{2s})$;

(ii) $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - \max\{t_2, t_3, t_5\}$, where $\alpha, \gamma \in [0, \frac{1}{2s})$;

(iii) $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - \max\{t_2, t_3, t_4, t_5\}$, where $\alpha, \beta, \gamma \in [0, \frac{1}{s})$;

Proof. (i), Let $F_S : \mathbb{R}^5 \to \mathbb{R}^+$. Define $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - t_5$, where $\gamma \in [0, \frac{1}{s})$. Then F_S satisfies an implicit relation.

$(FS1)$ F_S is non-increasing in the t_1 and t_5 variable;

$(FS2)$ for all $q, r \geq 0$, we have

$$\begin{aligned} F_S(q, r, r, , q, s(2q+r)) &= t_1 - \gamma t_5 \leq 0, \\ q - \gamma s(2q+r) &\leq 0, \\ (1-2s\gamma)q &\leq s\gamma r, \\ q &\leq \frac{s\gamma r}{(1-2s\gamma)}. \end{aligned} \tag{6.42}$$

Thus $q \leq \vartheta r$, with $\vartheta = \frac{s\gamma}{(1-2s\gamma)} < 1$.

$(FS3)$ $F_S(t, t, 0, 0, t) > 0$ for all $t > 0$.

$$\begin{aligned} F_S(t, t, 0, 0, t) &= t_1 - t_5 \leq 0, \\ u - s(2u+v) &\leq 0, \\ t - s(2t+t)) &\leq 0, \\ (1-3s)t &\leq 0, \\ t &\leq 0, \end{aligned}$$

which is a contradiction. Hence $F_S \in F_Q$ satisfies an implicit relation with $\gamma \in [0, \frac{1}{s})$. $\square$

The example 6.4.2 (ii, iii) can be proved similarly by following the above steps, to satisfy the implicit relation conditions imposed in Definition 6.8.

We prove the following theorem which is an extension of Theorem 6.2, from quasi-partial b-metric space to quasi-partial S_b-metric space setting. By using a pair of mapping.

Theorem 6.4 *Let (X, Sqp_b, s) be a complete quasi-partial S_b- metric space with $s \geq 1$, and let $\mathscr{A}, \mathscr{S} : X \to X$ be a pair of self-mappings. Assume that there exists $F_S \in F_Q$, satisfying $(FS1 - FS4)$ such that the following conditions hold:*

- *(a) there exists $\mathscr{A}X \subseteq \mathscr{S}X$ such that (X, qp_b) is complete,*
- *(b) there exists $u_0 \in X$ such that $\mathscr{S}u_n = \mathscr{A}u_{n-1}$,*
- *(c) $\mathscr{A}$ and $\mathscr{S}$ have a coincidence point in X,*
- *(d) $(\mathscr{A}, \mathscr{S})$ is nondencreasing and weakly compatible for some point u^* in X,*
- *(e) there exists an implicit function $F_S \in F_Q$ with*

$$F_S \left\{ \begin{array}{c} S_{qp_b}(\mathscr{A}u, \mathscr{A}u, \mathscr{A}v), S_{qp_b}(\mathscr{S}u, \mathscr{S}u, \mathscr{S}v), \\ S_{qp_b}(\mathscr{S}u, \mathscr{S}u, \mathscr{A}u), S_{qp_b}(\mathscr{S}v, \mathscr{S}v, \mathscr{A}v), \\ [S_{qp_b}(\mathscr{S}u, \mathscr{S}u, \mathscr{A}v) + S_{qp_b}(\mathscr{S}v, \mathscr{S}v, \mathscr{A}u)] \end{array} \right\} \leq 0, \tag{6.43}$$

$\forall u, v \in X$. Then $\mathscr{A}$ and $\mathscr{S}$ has a unique common fixed point.

Proof. Assume that $\mathscr{S}X \subseteq \mathscr{A}X$ and (X, S_{qp_b}) is a complete quasi-partial S_b-metric space, for u_0 with $(\mathscr{S}u_0, \mathscr{S}u_0, \mathscr{A}u_0,) \in X$, we construct a $\mathscr{S}$-$\mathscr{A}$-sequence $\{\mathscr{A}u_n\}$ with initial point u_0 satisfying

$$(\mathscr{S}u_0, \mathscr{S}u_0, \mathscr{A}u_0), (\mathscr{S}u_1, \mathscr{S}u_1, \mathscr{A}u_1), (\mathscr{S}u_2, \mathscr{S}u_2, \mathscr{A}x_2), \ldots, (\mathscr{S}u_{n+1}, \mathscr{S}u_{n+1}, \mathscr{A}u_{n+1})$$

$\forall n \in \mathbb{N}_{\mathbb{O}} = (\mathbb{N} \cup \{0\})$, thus, $\{\mathscr{A}u_n\}$, $\{\mathscr{S}u_n\} \in \mathscr{A}(X)$.

From assumption (b), let u_0 be an arbitrary element of X. If $\mathscr{S}u_0 = \mathscr{A}u_0$, then u_0 is a common fixed point of $\mathscr{A}$ and $\mathscr{S}$ and our proof is completed. Otherwise, if $\zeta x_0 \neq \eta u_0$, then $\mathscr{S}X \subseteq \mathscr{A}X$, now we choose $u_1 \in X$ such that $\mathscr{S}u_1 = \mathscr{A}u_0$. Again we can choose $u_2 \in X$ such that $\mathscr{S}u_2 = \mathscr{A}u_1$. Repeating this process the same way, we construct a sequence $\{\mathscr{S}u_n\} \subset X$, such that

$$\mathscr{S}u_{n+1} = \mathscr{A}u_n, \forall n \in \mathbb{N}_0.$$

If $\mathscr{S}u_{n-1} = \mathscr{S}u_n = \mathscr{A}u_{n-1}$, for all $n \geq 1$, then u_{n-1} is a coincidence point of $\mathscr{A}$ and $\mathscr{S}$ in X. Suppose that $\mathscr{S}u_{n-1} \neq \mathscr{S}u_n$ $\forall n \geq 1$. Then $S_{qp_b}(\mathscr{S}u_{n+1}, \mathscr{S}u_{n+1}, \mathscr{S}u_n) = S_{qp_b}(\mathscr{A}u_n, \mathscr{A}u_n, \mathscr{S}u_{n-1})$.

By taking $u = u_{n-1}$ and $v = u_n$ in (6.43), we have

$$F_S \left\{ \begin{array}{c} S_{qp_b}(\mathscr{A}u_{n-1}, \mathscr{A}u_{n-1}, \mathscr{A}u_n), S_{qp_b}(\mathscr{S}u_{n-1}, \mathscr{S}u_{n-1}, \mathscr{S}u_n), \\ S_{qp_b}(\mathscr{S}u_{n-1}, \mathscr{S}u_{n-1}, \mathscr{A}u_{n-1}), S_{qp_b}(\mathscr{S}u_n, \mathscr{S}u_n, \mathscr{A}u_n), \\ [S_{qp_b}(\mathscr{S}u_{n-1}, \mathscr{S}u_{n-1}, \mathscr{A}u_n) + S_{qp_b}(\mathscr{S}u_n, \mathscr{S}u_n, \mathscr{A}u_{n-1})] \end{array} \right\} \leq 0.$$

It follows that

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n+1}),S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_n),\\ S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_n),S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n+1}),\\ [S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_{n+1})+S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_n)]\end{array}\right\}\leq 0. \qquad (6.44)$$

By $(QPSb4)$ we have

$$\begin{aligned} S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_{n-1}) &\leq s[2S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n)+\\ &\quad S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1})]\\ &\quad -S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_n). \qquad (6.45)\end{aligned}$$

Using (6.45) in (6.43) we get

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n+1}),S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_n),\\ S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_n),S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n+1}),\\ [s[2S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n)+S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1})]\\ -S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_n)+S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_n)]\end{array}\right\}\leq 0.$$

Cosequently,

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n+1}),S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_n),\\ S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_n),S_{qp_b}(\mathscr{S}u_n,\mathscr{S}un,\mathscr{S}u_{n+1}),\\ s[2S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n)+S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1})]\end{array}\right\}\leq 0. \qquad (6.46)$$

By denoting $q=S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n)$ and $r=S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1})$ in (6.46) we get

$$F_S\left\{q,r,r,q,s(2q+r)\right\}\leq 0. \qquad (6.47)$$

By (6.47), in view of condition $(FS2)$ there exists $\vartheta\in[0,\frac{1}{s})$ and q is nonincreasing in the first variable, such that $uq\leq\vartheta r$, this implies that

$$S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n) \leq \vartheta S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1}); \qquad (6.48)$$

$\forall n\in\mathbb{N}$.

By induction in (6.48), we get

$$\begin{aligned} S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n) &\leq \vartheta S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1}),\\ &\leq \vartheta^2 S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}x_{n-2}),\\ &\leq \dots\\ &\leq \vartheta^n S_{qp_b}(\mathscr{S}u_0,\mathscr{S}u_0,\mathscr{S}u_1). \qquad (6.49)\end{aligned}$$

Therefore, $\lim_{n\to\infty}S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n)=0$.

Now, we prove that $S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{S}u_n)$ is a Cauchy sequence. Let $n,m\in\mathbb{N}$, for any positive intergers such that $n>m$, using $(QPSb4)$ we have

$$\begin{aligned}
S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_m) &\leq s[2S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1})+S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_m)]\\
&\quad -S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_{n-1}),\\
&= 2sS_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n-1})\\
&\quad +2s^2S_{qp_b}(\mathscr{S}u_{n-1},\mathscr{S}u_{n-1},\mathscr{S}u_{n-2})\\
&\quad +s^2S_{qp_b}(\mathscr{S}u_{n-2},\mathscr{S}u_{n-2},\mathscr{S}u_m)+\\
&\quad \cdots+s^{m-n-1}S_{qp_b}(\mathscr{S}u_{m+1},\mathscr{S}u_{m+1},\mathscr{S}u_m)\\
&\leq 2[s\vartheta^{n-1}+s^2\vartheta^{n-2}+s^3\vartheta^{n-3}+\\
&\quad \cdots+s^{m-n+1}\vartheta^{m}]S_{qp_b}(\mathscr{S}u_0,\mathscr{S}u_0,\mathscr{S}u_1),\\
&\leq 2s\vartheta^{n-1}[1+s\vartheta+s^2\vartheta^2+\\
&\quad \cdots+s^{m-1}\vartheta^{m-n+1}]S_{qp_b}(\mathscr{S}u_0,\mathscr{S}u_0,\mathscr{S}u_1),\\
&\leq \frac{2s\vartheta^{n-1}}{1-s\vartheta}S_{qp_b}(\mathscr{S}u_0,\mathscr{S}u_0,\mathscr{S}u_1). \qquad (6.50)
\end{aligned}$$

Since $\vartheta\in[0,\frac{1}{s})$, we conclude that $\frac{2s\vartheta^{n-1}}{1-s\vartheta}S_{qp_b}(\mathscr{S}u_0,\mathscr{S}u_0,\mathscr{S}u_1)\to 0$ as $n\to\infty$. Therefore, $\{\mathscr{S}u_n\}$ is a Cauchy sequence in $\mathscr{S}(X)$. Thus $S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_m)\to 0$ as $n,m\to\infty$.

Similarly, suppose that $\mathscr{A}X\subseteq\mathscr{S}X$. For every $u_0\in X$ we consider the sequence $\{\mathscr{A}u_n\}\in X$ defined by

$$\begin{aligned}
\mathscr{S}u_n &= \mathscr{A}u_{n-1},\\
\mathscr{S}u_{n+1} &= \mathscr{A}u_n.
\end{aligned}$$

If $\mathscr{S}u_{n+1}=\mathscr{A}u_n$, then u_n is a fixed point of $\mathscr{S}$ and $\mathscr{A}$ and the proof is completed. On contrary, assume that $\mathscr{S}u_{n+1}\neq\mathscr{A}u_n$ and $u_{n+1}\neq u_n$. Then, $u=u_n$ and $v=u_{n+1}$ in Equation (6.43) we have

$$F_S\left\{\begin{array}{c}
S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1}),S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}u_{n+1}),\\
S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{A}u_n),S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{A}u_{n+1}),\\
[S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{A}u_{n+1})+S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{S}u_{n+1},\mathscr{A}u_n)]
\end{array}\right\}\leq 0. \qquad (6.51)$$

By substituting $\mathscr{S}u_n=\mathscr{A}u_{n-1}$ and $\mathscr{S}u_{n+1}=\mathscr{A}u_n$ in Equation (6.51), we get

$$F_S\left\{\begin{array}{c}
S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\zeta x_{n+1}),S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_n),\\
S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_n),S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1}),\\
[S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_{n+1})+S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_n)]
\end{array}\right\}\leq 0. \qquad (6.52)$$

By $(QPSb4)$, we have

$$\begin{aligned}
S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_{n+1}) &\leq s[2S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_n)+\\
&\quad S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1})]\\
&\quad -S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_n). \qquad (6.53)
\end{aligned}$$

Using (6.53) in (6.52) we get

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1}),S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_n),\\ S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_n),S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1}),\\ [s[2S_{qp_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_n)+S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1})]\\ -S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_n)+S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_n)] \end{array}\right\}\leq 0. \qquad (6.54)$$

Since quasi-partial S_b is not symmetry, by $(FS2)$, we reach in similar results from the right hand side of Cauchy convengence.

Using $(QPSb4)$ and $(FS1)$, since is a non-decreasing in the fifth variable, and satisfy

$$q \ \leq \ \vartheta r,$$

where $\vartheta \in [0,\frac{1}{s})$.

Which implies that

$$\begin{aligned} S_{qs_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1}) &\leq \vartheta S_{qs_b}(\mathscr{A}u_{n-1},\mathscr{A}u_{n-1},\mathscr{A}u_n),+\\ &\quad \cdots+\\ &\leq \vartheta^n S_{qs_b}(\mathscr{A}u_0,\mathscr{A}u_0,\mathscr{A}u_1). \qquad (6.55)\end{aligned}$$

For $n\to\infty$ in (6.55), leads to $S_{qs_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1})\to 0$.

Using $(QPSb4)$, for all $n,m\in\mathbb{N}_0$ with $m>n$, we obtain

$$\begin{aligned} S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_m) &\leq s[2S_{qp_b}(\mathscr{S}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1})+\\ &\quad S_{qp_b}(\mathscr{S}u_{n+1},\mathscr{A}u_{n+1},\mathscr{A}u_m)]\\ &\quad -S_{qp_b}(\mathscr{A}u_{n+1},\mathscr{A}u_{n+1},\mathscr{A}u_{n+1}),\\ &= 2sS_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_{n+1})+\\ &\quad 2s^2S_{qp_b}(\mathscr{A}u_{n+1},\mathscr{A}u_{n+1},\mathscr{A}u_{n+2})\\ &\quad +s^2S_{qp_b}(\mathscr{A}u_{n+2},\mathscr{A}u_{n+2},\mathscr{A}u_m)+\\ &\quad \cdots+s^{m-n-1}S_{qp_b}(\mathscr{A}u_{m-1},\mathscr{A}u_{m-1},\mathscr{A}u_m)\\ &\leq 2[s\vartheta^n+s^2\vartheta^{n+1}+s^2\vartheta^{n+2}+\cdots+\\ &\quad s^{m-n-1}\vartheta^{m-1}]S_{qp_b}(\mathscr{A}u_0,\mathscr{A}u_0,\mathscr{A}u_1),\\ &= 2s\vartheta^n[1+s\vartheta+s^2\vartheta^2+\\ &\quad \cdots+s^{m-n-2}\vartheta^{m-n-1}]S_{qp_b}(\mathscr{A}u_0,\mathscr{A}u_0,\mathscr{A}u_1),\\ &\leq \frac{2s\vartheta^n}{1-s\vartheta}S_{qp_b}(\mathscr{A}u_0,\mathscr{A}u_0,\mathscr{A}u_1). \qquad (6.56)\end{aligned}$$

Since $\vartheta\in[0,\frac{1}{s})$, we conclude that $\frac{2s\vartheta^n}{1-s\vartheta}S_{qp_b}(\mathscr{A}u_0,\mathscr{A}u_0,\mathscr{A}u_1)\to 0$ as $n\to\infty$. Therefore, $\{\mathscr{A}u_n\}$ is a Cauchy sequence in $\mathscr{A}(X)$. Thus $S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}u_m)\to 0$ as $n,m\to\infty$.

Now we show that u^* is a fixed point of $\mathscr{A}u$ such that $u^*=\mathscr{A}u^*$
and $\lim_{n,m\to\infty}S_{qp_b}(u_n,u_n,\mathscr{A}u^*)=S_{qp_b}(u^*,u^*,\mathscr{A}u^*)=0$. Let $u=u_n$ and $v=u^*$, using

$(QPSb4)$ we obtain

$$\begin{aligned} Sqp_b(u^*,u^*,\mathscr{A}u^*) &\leq s[Sqp_b(u^*,u^*,u_{n+1})+Sqp_b(u^*,u^*,u_{n+1})+ \\ &\quad Sqp_b(\mathscr{A}u^*,\mathscr{A}u^*,u_{n+1})]- \\ &\quad Sqp_b(u_{n+1},u_{n+1},u_{n+1}), \\ &= s[2Sqp_b(u^*,u^*,u_{n+1})+Sqp_b(\mathscr{A}u^*,\mathscr{A}u^*,u_{n+1})]- \\ &\quad Sqp_b(u_{n+1},u_{n+1},u_{n+1}). \end{aligned} \tag{6.57}$$

Taking the limit $n \to \infty$ in (6.57), we get

$$\begin{aligned} Sqp_b(u^*,u^*,\mathscr{A}u^*) &\leq s[2Sqp_b(u^*,u^*,u^*)+Sqp_b(\mathscr{A}u^*,\mathscr{A}u^*,u^*)]- \\ &\quad Sqp_b(u^*,u^*,u^*), \\ &= s[0+Sqp_b(\mathscr{A}u^*,\mathscr{A}u^*,u^*)]-0, \\ &\leq sSqp_b(\mathscr{A}u^*,\mathscr{A}u^*,u^*), \end{aligned}$$

which is a contradiction. Hence, $u^*=\mathscr{A}u^*$. Thus u^* is a fixed point of ζ.

From Definition 6.5, we show that u^* is a coincidence point of $\mathscr{A}$ and $\mathscr{S}$. Since $\mathscr{A}X$ is complete there exists $u^*, v^* \in X$ such that $u^*=\mathscr{S}y^*$. Which implies that

$$\lim_{n\to\infty}\mathscr{A}u_n=\lim_{n\to\infty}\mathscr{S}u_n=\mathscr{S}v^*=u^*. \tag{6.58}$$

By taking $u=x_n$ and $v=v^*$ in (6.43), we obtain

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{A}u_n,\mathscr{A}u_n,\mathscr{A}v^*),S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{S}v^*), \\ S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{A}u_n),S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{A}v^*), \\ {[S_{qp_b}(\mathscr{S}u_n,\mathscr{S}u_n,\mathscr{A}v^*)+S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{A}u_n)]} \end{array}\right\}\leq 0. \tag{6.59}$$

Letting $n\to\infty$ in (6.59) and using (6.58), we get

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{A}v^*),S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{S}v^*), \\ S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{A}v^*),S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{A}v^*), \\ {[S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{A}v^*)+S_{qp_b}(\mathscr{S}v^*,\mathscr{S}v^*,\mathscr{A}v^*)]} \end{array}\right\}\leq 0, \tag{6.60}$$

by assumption $(FS2)$ and continuity of F_S, we obtain $S_{qp_b}(\mathscr{A}v^*,\mathscr{A}v^*,\mathscr{S}v^*)\leq 0$.

Consequently,

$$\eta v^*=\mathscr{A}v^*=u^*.$$

Thus, u^* is a coincidence point of $\mathscr{S}$ and $\mathscr{A}$.

Now, we assume that $\mathscr{S}$ and $\mathscr{A}$ are either $\mathscr{S}$ or $\mathscr{A}$-weakly compatable. Let $\lim_{n\to\infty}u_n=u^*,\mathscr{S}u_n=\mathscr{S}u^*,\mathscr{A}u_n=\mathscr{A}u^*,\mathscr{S}\mathscr{S}u^*=\mathscr{S}\mathscr{A}u^*$ and $\mathscr{S}\mathscr{A}u^*=\mathscr{A}\mathscr{S}u^*$.

Suppose $u=\mathscr{S}u^*$ and $v=v^*$, using (6.43) and definition 6.5, we get

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{A}\mathscr{S}u^*,\mathscr{A}\mathscr{S}u^*,\mathscr{A}u^*),S_{qp_b}(\mathscr{S}\mathscr{S}u^*,\mathscr{S}\mathscr{S}u^*,\mathscr{S}u^*), \\ S_{qp_b}(\mathscr{S}\mathscr{S}u^*,\mathscr{S}\mathscr{S}u^*,\mathscr{A}\mathscr{S}u^*),S_{qp_b}(\mathscr{S}u^*,\mathscr{S}u^*,\mathscr{A}u^*), \\ {[S_{qp_b}(\mathscr{S}\mathscr{S}u^*,\mathscr{S}\mathscr{S}u^*,\mathscr{A}u^*)+S_{qp_b}(\mathscr{S}u^*,\mathscr{S}u^*,\mathscr{A}\mathscr{S}u^*)]} \end{array}\right\}\leq 0, \tag{6.61}$$

yields to,

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathscr{A}\mathscr{S}u^*,\mathscr{A}\mathscr{S}u^*,\mathscr{A}u^*),S_{qp_b}(\mathscr{S}\mathscr{A}u^*,\mathscr{S}\mathscr{A}u^*,\mathscr{S}u^*),\\ S_{qp_b}(\mathscr{S}\mathscr{A}u^*,\mathscr{S}\mathscr{A}u^*,\mathscr{A}\mathscr{S}u^*),S_{qp_b}(\mathscr{S}u^*,\mathscr{S}u^*,\mathscr{A}u^*),\\ {[S_{qp_b}(\mathscr{S}\mathscr{A}u^*,\mathscr{S}\mathscr{A}u^*,\mathscr{A}u^*)+S_{qp_b}(\mathscr{S}u^*,\mathscr{S}u^*,\mathscr{A}\mathscr{S}u^*)]} \end{array}\right\}\leq 0. \quad (6.62)$$

Which implies that

$$S_{qp_b}(\mathscr{A}u^*,\mathscr{A}u^*,\mathscr{S}\mathscr{A}u^*)\leq 0.$$

We have $\mathscr{S}u^*=\mathscr{S}\mathscr{A}u=\mathscr{A}\mathscr{S}u=\mathscr{A}u^*$. Thus, $\mathscr{S}$ and $\mathscr{A}$ are weakly compatible self maps of a set X. Therefore, $\mathscr{S}$ and $\mathscr{A}$ have a unique point of coincidence $u^*=\mathscr{S}u=\mathscr{A}u$, then u^* is the unique common fixed point of $\mathscr{S}$ and $\mathscr{A}$. □

Pathak *et al.* (2007), in their work, considered an example in which weakly compatible mapping is not compatible. In this work, we use one more example of this type, which satisfies quasi partial S_b-metric space and use it to formulate an implicit function that satisfies all conditions imposed in Definition 6.5 and Theorem 6.4.

Example 6.5 *Consider* $X=[0,\infty]$ *endowed with complete quasi-partial* S_b *metric space, defined by metric* $S_{qp_b}(u,u,v)=2(u-v)^2$ *on* X. *Define a pair of mappings* $\mathscr{A},\mathscr{S}:X\to X$ *by*

$$\mathscr{S}u=\begin{cases}\cos u & \textit{if } u\neq 1\\ 0 & \textit{if } u=1,\end{cases}$$

and

$$\mathscr{A}u=\begin{cases}e^u & \textit{if } u\neq 1\\ 0 & \textit{if } u=1,\end{cases}$$

by Definition 6.5, it obvious that at $u=0$, we have $u^*=\mathscr{S}u=\mathscr{S}\mathscr{S}u=\mathscr{A}\mathscr{S}u=\mathscr{A}u$, then $u^*=0$ is the unique common fixed point of $\mathscr{S}$ and $\mathscr{A}$. Therefore, the mappings $\mathscr{S}$ and $\mathscr{A}$ are weakly compatible. Define continuous function $F:\mathbb{R}^5_+\to\mathbb{R}$ by

$$F(t_1,t_2,t_3,t_4,t_5) \;=\; t_1-\gamma t_5.$$

i.e.,

$$F_S(u,v,v,u,s(2u+v)) \;=\; t_1-\gamma t_5.$$

With a view to verify assumptions (a) and (d) of Theorem 6.4. Consider $\mathscr{A}u,\mathscr{S}u\in X$ so that

$$\begin{aligned} t_1 &\leq \gamma t_5.\\ S_{qp_b}(\mathscr{A}u,\mathscr{A}u,\mathscr{S}v) &\leq \gamma[s[2S_{qp_b}(\mathscr{A}u,\mathscr{A}u,\mathscr{S}v)+S_{qp_b}(\mathscr{A}v,\mathscr{A}v,\mathscr{S}u)]].(6.63)\end{aligned}$$

Recall, the quasi partial S_b-metric as,

$$\begin{aligned} S_{qp_b}(\mathscr{A}u,\mathscr{A}u,\mathscr{S}v) &= 2(\mathscr{A}u-\mathscr{S}v)^2, \\ &= 2(e^u-\cos v)^2. \end{aligned} \tag{6.64}$$

Similarly,

$$\begin{aligned} S_{qp_b}(\mathscr{A}v,\mathscr{A}v,\mathscr{S}u) &= 2(\mathscr{A}v-\mathscr{S}u)^2 \\ &= 2(e^v-\cos u)^2. \end{aligned} \tag{6.65}$$

Using (6.64) and (6.65) in (6.63), we get

$$\begin{aligned} 2(e^u-\cos v)^2 &\leq \gamma s[2(e^u-\cos v)^2+2(e^v-\cos u)^2], \\ 2(e^u-\cos v)^2(1-2\gamma s) &\leq \gamma s[4(e^v-\cos u)^2], \\ 2(e^u-\cos v)^2 &\leq \frac{\gamma s}{(1-2\gamma s)}[2(e^v-\cos u)^2], \end{aligned} \tag{6.66}$$

which means

$$q \leq \vartheta r.$$

Hence, F_S satisfies $FS1, FS2$ and $FS3$ for $\vartheta \in [0,\frac{1}{s}]$. Also, all assumptions of Theorem 6.4 and Definition 6.5 are satisfied. It is observed that the pair $(\mathscr{S},\mathscr{A})$ has a common fixed point. Thus, they adimit a coincidence fixed point.

6.5 Some Applications

In this section, we have four subsections. In subsection 6.5.1, we prove the existence of the solution for two boundary value second order differential equations via implicit mapping with binary relation in metric-like space. Next in subsection 6.5.2, we investigate the existence of the solution to the Volterra-Fredholm type integral equation, which is used to illustrate the use of Theorem 6.3 for the existence of a common fixed point of a pair of maps in a metric like space.

In subsection 6.5.3, we discuss the existence solution of the two-point boundary value problem of the second-order differential equation in quasi partial S_b-metric space.

In subsection 6.5.4, we prove the existence solution for Caputo type nonlinear fractional differential equations as applications to utilise the results obtained in Theorem 6.4 were a common solution applied in quasi partial S_b-metric space setting.

6.5.1 Existence Solution for a Second-Order Differential Nonlinear Two Boundary Value Problem

In this subsection, we consider a second-order differential non-linear two boundary value problem. The following problem is motivated by Hunter and Nachtergaele (2001), Agarwal *et al.* (2001) and Nashine *et al.* (2018).

$$\begin{cases} u^{''}(t) = f(t,u(t),u'(t)),\ t \in (0,1), \\ u(a) = u_1, \\ u(b) = u_2,\ a,b \in [0,1], \end{cases} \tag{6.67}$$

where $f : [0,1] \times X \times X \longrightarrow X$ is a continuous function.

This problem is equivalent to the integral equation

$$u(t) \quad = \quad h(t) + \int_a^b G(t,s) f(s,u(s),u'(s))ds, \forall\, t,s \in [a,b], \tag{6.68}$$

where the Green's function associated with the above integral equation is given by

$$G(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{b-a}, & a \leq s \leq t \leq b, \\ \dfrac{(b-s)(t-a)}{b-a}, & a \leq t \leq s \leq b, \end{cases}$$

and $h(t)$ satisfies $h'' = 0$, $h(a) = u_1$, $h(b) = u_2$.

By Theorem 6.3 (a), $TX \subseteq SX$. The fixed point of S is also a solution of (6.67).

Now we prove our results by establishing the existence of a common fixed point for a pair of self mappings:

Theorem 6.5 Let $T,S : C([a,b]) \longrightarrow C([a,b])$ be self maps of a metric-like space (X,σ) such that the following conditions hold:

(i) f: $[0,1] \times X \times X \longrightarrow X$ is a nonincreasing function in the fifth and sixth variable,

(ii) There exist a functions $f : [0,1] \times X \times X \longrightarrow X$ with constants α and β such that

$$|f(t,u(t),v'(t))| - |f(t,u(t),v'(t))| \quad \leq \quad L|\alpha|u-v| + \beta|u'-v'||,$$

for all $t \in [0,1]$ and $u,v \in C^1([a,b],X)$,

(iii) there exists a path $a, b \in [0,1]$ and $\alpha, \beta > 0$ such that

$$k = \frac{\alpha + 4\beta}{8}, \text{ and } Lk \leq 1.$$

Then, the non linear integral equation has a common solution in $C^1([a,b],X)$ and (6.68) has a solution. Also it is the solution of differential equation (6.67).

Proof: Consider $C^1([a,b],X)$ with the metric

$$\sigma(x,y) = \alpha \sup_{t\in[a,b]} |u-v| + \beta \sup_{t\in[a,b]} |u'-v'|.$$

The (X, σ) is a complete metric-like space.

Let $T, S : X \longrightarrow X$ be two operators defined as

$$Tu(t) \;=\; h(t) + \int_a^b G(t,s) f(s,u(s),u'(s))ds, \forall\, t,s \in [a,b],$$

and

$$Sv(t) \;=\; h(t) + \int_a^b G(t,s) f(s,v(s),v'(s))ds, \forall\, t,s \in [a,b],$$

where f and h are continuous functions. Now, u is a solution of (6.68) if and only if u is a common fixed point of T and S. Since T and S are increasing in the fifth and sixth variables and other assertion of Theorem 6.3 are satisfied, by using condition (ii) of Theorem 6.5 we obtain

$$\begin{aligned}
|Tu(t) - Sv(t)| &= \int_a^b |G(t,s)||f(s,u(s),u'(s)) - f(s,v(s),v'(s))|ds, \\
&= \int_a^b |G(t,s)|ds(\alpha|u-v| + \beta|u'-v'|), \\
&= L\sigma(u,v)\int_a^b |G(t,s)|ds.
\end{aligned}$$

For each $a, b \in [0,1]$, we have

$$\int_a^b G(t,s)ds \;=\; \max_{a\leq t\leq b} \frac{(b-t)(t-a)}{8} = \frac{(b-a)^2}{8},$$

$$\alpha \sup_{t\in[a,b]} |u(t) - v(t)| \;=\; \alpha\frac{(b-a)^2}{8}\sigma(u,v) = \frac{\alpha}{8}\sigma(u,v), \tag{6.69}$$

Similarly,

$$\begin{aligned}|Tu'(t)-Sv'(t)| &= \int_a^b |G(t,s)||f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds,\\ &= L\int_a^b |G(t,s)|ds(\alpha|u(t)-v(t)|+\beta|u'(t)-v'(t)|),\\ &= L\sigma(u,v)\int_a^b |G(t,s)|ds,\end{aligned}$$

$$\int_a^b G(t,s)ds = \max_{a\le t\le b}\frac{(b-t)^2+(t-a)^2}{2(b-a)} = \frac{b-a}{2},$$

$$\beta\sup_{t\in[a,b]}|u'-v'| = \beta\frac{(b-a)}{2}\sigma(u',v') = \frac{\beta}{2}\sigma(u',v'), \tag{6.70}$$

From adding (6.69) and (6.70) we obtain

$$\begin{aligned}\sigma(Tu,Sv) &\le \frac{(b-a)^2}{8}\sigma(u,v)+\frac{(b-a)}{2}\sigma(u,v),\\ &\le \left[\frac{\alpha}{8}+\frac{\beta}{2}\right]\sigma(u,v).\end{aligned}$$

Since

$$k = \frac{\alpha+4\beta}{8} \textit{ and } Lk<1,$$

we have

$$\sigma(Tu,Sv) \le L\sigma(u,v).$$

Therefore $u\in X$, hence u is a common fixed of T and S, also a solution to integral equation (6.68). Thus, a differential equation (6.67) has a solution.

6.5.2 Existence of the Solution of Volterra-Fredholm Type Integral Equation

Now, in this subsection, we investigate the existence of the solution to the Volterra-Fredholm type integral equation, which is used to illustrate the use of Theorem 6.3 for the existence of a common fixed point of a pair of maps in metric space. The following integral equation inspired by Nashine *et al.* (2018) and Agarwal *et al.* (2018). The equation arise from the theory of parabolic boundary valued problems, which is the mathematical modellings of the spatio-temporal development of epidemic and various physical and biological models.

$$u(t,x) \;=\; h(t,x)+\int_0^t\int_{\mathbb{R}^2} K(t,x,s,y,u(s,y))dyds, \forall\, t,x\in D, \qquad (6.71)$$

where $h : D \to \mathbb{R}^{\mathbb{N}}$, K : D $\times D \to \mathbb{R}^{\mathbb{N}}$, $D = [0,T]\times\Omega, T > 0$ and $\Omega = \mathbb{R}^{\mathbb{N}}$ is the non empty and closed set of Euclidean space $\mathbb{R}^{\mathbb{N}}$ equipped with norm $\|.\|$, $\forall\, \mathbb{N} \geq 1$.

Let $(X,\|.\|)$ be a Banach space. Define the mapping $\sigma : X\times X \to [0,\infty)$ by

$$\sigma(x,y) = \|x-y\|.$$

Then (X,σ) is a complete metric-like space.

By Theorem 6.3 (a), $TX \subseteq SX$. The fixed point of S is also a solution of (6.71).

Now we prove our results by establishing the existence of a common fixed point for a pair of self mappings:

Theorem 6.6 Let $T,S : C^{\mathbb{N}}([a,b]) \longrightarrow C^{\mathbb{N}}([a,b])$ be self maps of a metric-like space (X,σ). Suppose the following assumptions hold:

(i) the function $h : D \to \mathbb{R}_+$ and $K : D\times D\times\mathbb{R}_+ \longrightarrow X$ are continuous,

(ii) there exists a continuous function $L : D\times D \to [0,\infty)$ such that

$$\|K(t,x,s,y,u(s,y)) - K(t,x,s,y,v(s,y))\| \;\leq\; L(t,x,s,y)\|u-v\|,$$

for all $t,x,s,y,u(s,y) \in D\times D\times\mathbb{R}^{\mathbb{N}}$,

(iii) there exists a path $a,b\in[0,1]$ with a constant $\gamma\in[0,1)$ such that

$$\int_0^t\int_{\mathbb{R}^2} L(t,x,s,y)\|u-v\|dyds \leq \frac{1+t}{6+7t^2}[\ln(1+\frac{1}{3}|x|) - \ln(1+\frac{1}{3}|x|)],$$

where

$$L(t,x,s,y) = \gamma \;=\; \frac{1+t}{6+7t^2} < 1.$$

Then, the Volterra-Fredholm integral equation (6.71) has a unique common solution in $C^{\mathbb{N}}([a,b],X)$.

Proof: Consider $C^{\mathbb{N}}([a,b],X)$ with the metric

$$\sigma(x,y) = \|x-y\|.$$

The (X,σ) is a complete metric-like space.

Let $T,S : X \to S$ be two operators such that S$\in X$ and $u \in S$. Define

$$Tu(t,x) \quad = \quad h(t,x) + \int_0^t \int_{\mathbb{R}^2} K(t,x,s,y,u(s,y))dyds, \forall\, t,x \in D. \tag{6.72}$$

Since $TX \subset SX$,we prove that T maps S into itself. So, suppose that $Tu : [0,T] \times \mathbb{R} \to \mathbb{R}^2$ is continuous mapping. Now, on contrary to that, we claim that $T : S \to S$ is not a contraction. So, let (u,v) be a pair of elements in S. For all $(t,x) \in D$ and using condition (ii) of Theorem 6.7 we get

$$\begin{aligned}
\|Tu - Sv\| &= \int_0^t \int_{\mathbb{R}^2} \|K(t,x,s,y,u(s,y)) - K(t,x,s,y,v(s,y))\| dyds, \\
&\leq \int_0^t \int_{\mathbb{R}^2} L(t,x,s,y)\|u-v\| dyds, \\
&\leq \frac{1+t}{6+7t^2}\left[\ln(1+\frac{1}{3}\|u\|) - \ln(1+\frac{1}{3}\|v\|) \right], \\
&\leq \frac{1+t}{6+7t^2} \ln \left[\frac{1+\frac{1}{3}\|u\|}{1+\frac{1}{3}\|u\|} \right], \\
&\leq \frac{1+t}{6+7t^2} \ln \left[1 + \frac{\frac{1}{3}\|u\| - \frac{1}{3}\|v\|}{1+\frac{1}{3}\|v\|} \right], \\
&\leq \gamma\|u-v\|, \\
\|Tu - Sv\| &\leq \gamma\|u-v\|, \\
\sigma(Tu,Sv) &\leq \gamma\sigma(u,v),
\end{aligned}$$

which is a contradiction. Hence u is a common fixed of T and S, also a solution to integral equation (6.71).

6.5.3 Existence Solution of the Two Boundary Value Second Order Differential Equation in Quasi S_b-Metric Space

In this subsection we discuss the existence of a solution of the boundary value problem by considering a space to be quasi partial S_b metric space. Now, we consider the two point boundary value problem of the second order differential equation. The following example is motivated by Edwards (1965), Yan *et al.* (2012), Pathak (2018) and Borisut *et al.* (2019).

$$\begin{cases} u^{''}(t) = f(t,u(t),u'(t)),\ 0 \le t \le T, \\ u(0) = \alpha, \\ u(T) = \beta, \end{cases} \tag{6.73}$$

where $T > 0$ and $f : [0,T] \times X \times X \longrightarrow X$ is a continuous function.

This boundary value problem is equivalent to the integral equation

$$u(t) \quad = \quad \alpha + \frac{\beta - \alpha}{T} t + \int_0^T G(t,s) f(s,u(s),u'(s)) ds, \forall\, t,s \in [0,T]. \tag{6.74}$$

where the Green's function associated with the above integral equation is given by

$$G(t,s) = \begin{cases} \dfrac{s(T-t)}{T}, & 0 \le s \le t \le T, \\ \\ \dfrac{t(T-s)}{T}, & 0 \le t \le s \le T, \end{cases}$$

and $\alpha, \beta > 0$.

We prove our results by establishing a common fixed point for a pair of weakly compatible self mappings in quasi-partial S_b metric space.

Theorem 6.7 *Let $\mathscr{A}, \mathscr{S} : C([0,T]) \longrightarrow C([0,T])$ be self maps of a quasi-partial S_b metric space (X, S_{qp_b}) such that the following conditions hold:*

(i) *there exists $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function and η-weakly increasing in the first and fifth variables with $\gamma \in [0, \frac{1}{s}]$ such that*

$$|f_1(t,u(t),u'(t))| - |f_2(t,v(t),v'(t))| \quad \le \quad \gamma \sqrt{\frac{\ln[(u-v)^2+1]}{u-v}},$$

where $|u(s) - v(s)| = \gamma \sqrt{\frac{\ln[(u-v)^2+1]}{u-v}}$ and for increasing of u and v, we have $u, v \in C^1([0,T],X)$,

(ii) *the Green's function is given by*

$$\int_0^T G(t,s) \le \frac{1}{8}.$$

Then, the integral equation (6.74) *has a common solution in* $C^1([0,T],X)$.

Proof: Let $C^1([0,T],X) = f : [0,T] \to \mathbb{R}$ is a continuous function. Now, we define the function $S_{qp_b} : C[0,T] \times C[0,T] \times C[0,T] \to [0,\infty)$ with the quasi-partial S_b metric

$$S_{qp_b}(u,u,v) = 2\left(\sup_{t\in[0,T]} |u(t)-v(t)|\right)^2 + 2\left(\sup_{t\in[0,T]} |u'(t)-v'(t)|\right)^2.$$

Then, (X, S_{qp_b}) is a complete quasi-partial S_b metric space.

Let $\mathscr{A}, \mathscr{S} : X \longrightarrow X$ be two $\mathscr{S}$-weakly compatible operators defined by

$$\mathscr{A}u(t) = \alpha + \frac{(\beta-\alpha)t}{T} + \int_0^T G(t,s) f_1(t,s,u(s),u'(s))ds, \forall\, t,s \in [0,T].$$

and

$$\mathscr{S}v(t) = \alpha + \frac{(\beta-\alpha)t}{T} + \int_0^T G(t,s) f_2(t,s,v(s),v'(s))ds, \forall\, t,s \in [0,T].$$

Where f_1, f_2 and α, β are continuous functions.

Now, u^* is a solution of (6.74) if and only if u^* is a common fixed point of $\mathscr{A}$ and $\mathscr{S}$, since $\mathscr{A}$ and $\mathscr{S}$ are increasing in the first and fifth variables and other assertion of Theorem 6.7 are satisfied. We show that $\mathscr{A}$ and $\mathscr{S}$ are contractions in X.

For each $t \in [0,1]$, by (ii), we have

$$\int_a^b G(t,s)ds = \frac{1}{2}t(t-1).$$

and sup-norm of $t(1-t) = \frac{1}{4}$, therefore

$$\sup_{t\in[a,b]} \int_a^b G(t,s)ds = \frac{1}{8}.$$

By using condition (i) of Theorem 6.7, we discuss the following cases:

Case I.

$$\begin{aligned} |\mathscr{A}u(t) - \mathscr{S}v(t)| &= \int_0^T |f_1(s,u(s),u'(s)) - f_2(s,v(s),v'(s))|ds, \\ &\leq 2\left(\int_0^T |G(t,s)|ds|u(s)-v(s)|\right)^2, \\ &\leq 2\left(\frac{\gamma}{8}\sqrt{\frac{\ln[(|u-v|)^2+1]}{|u-v|}}\right)^2. \end{aligned} \tag{6.75}$$

Case II.

$$\begin{aligned}
|\mathscr{A}u'(t)-\mathscr{S}v'(t)| &= \int_0^T |f_1(s,u(s),u'(s))-f_2(s,v(s),v'(s))|ds, \\
&\leq 2\left(\int_0^T |G(t,s)|ds|u'(s)-v'(s)|\right)^2, \\
&\leq 2\left(\frac{\gamma'}{8}\sqrt{\frac{\ln[(|u'-v'|)^2+1]}{|u'-v'|}}\right)^2. \qquad (6.76)
\end{aligned}$$

By combining (6.75) and (6.76) , we obtain

$$\begin{aligned}
|\mathscr{A}u(t)-\mathscr{S}v(t)|+|\mathscr{A}u'(t)-\mathscr{S}v'(t)| &\leq 2\left(\frac{\gamma}{8}\sqrt{\frac{\ln[(|u-v|)^2+1]}{|u-v|}}\right)^2 + \\
& \quad 2\left(\frac{\gamma'}{8}\sqrt{\frac{\ln[(|u'-v'|)^2+1]}{|u'-v'|}}\right)^2. \\
S_{qp_b}(\mathscr{A}u,\mathscr{A}u,\mathscr{S}v) &\leq \vartheta S_{qp_b}(u,u,v).
\end{aligned}$$

Therefore $u^* \in X$ is a common fixed of $\mathscr{A}$ and $\mathscr{S}$, also a solution to integral equation (6.74). Hence the differential equation (6.73) has a solution.

6.5.4 Existence of a Common solution of Weakly Compatible Mappings for Nonlinear Fractional Differential Equation in Quasi-Partial S_b Metric Space

The purpose of this subsection is to provide an application of Theorem 6.4 to get a common solution of weakly compatible mappings for a nonlinear fractional differential equation, where we can apply a generalised mapping in quasi partial S_b metric spaces.

We investigate the existence of a unique common fixed point for η-weakly compatible mappings of the Caputo derivative with the fractional order of the nonlinear fractional differential equation.

This form of fractional derivative for a continuous function $f:[0,\infty)\to\mathbb{R}$ is given by Abdeljawad *et al.* (2019) and Zahed *et al.* (2020) as: Caputo fractional derivative of $\eta(t)$ order $\alpha>0$ is denoted by ${}^{C}\mathscr{D}^{\alpha}_{\eta}(t)$ and defined as

$${}^{C}\mathscr{D}^{\alpha}\eta(t)=\frac{1}{\Gamma(i-\alpha)}\int_0^t (t-\tau)^{i-\alpha-1}\eta^i(\tau)d\tau,$$

with $i = [\alpha] + 1 \in \mathbb{N}$, where $\alpha \in [i-1, i]$ and $[\alpha]$ denotes the greatest integers of α (i.e., the greatest part of α) and $\alpha : [0, \infty) \to \mathbb{R}$ is a continuous function.

We denote $X = C([0,1], \mathbb{R})$ the set of all continuous functions from $[0,1]$ into $\mathbb{R}$.

The Caputo fractional differential equation has several applications in mathematics, i.e., in image processing, Digital data processing, electrical signal, acoustics, physics and probability theory (one can see in (Zölzer *et al.* 2002)). The following nonlinear fractional differential equation is inspired by Baleanu *et al.* (2013), Karapinar *et al.* (2019), Kanwal *et al.* (2019) and Budhia *et al.* (2020).

Consider the following nonlinear fractional differential equation.

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}u(t) = f(t, u(t)),\ t \in (0,1), 1 < \alpha \leq 2, \\ \\ u(0) = 0, u(1) = \int_0^{\sigma} u(\tau)d\tau\ (0 < \sigma < 1) \end{cases} \tag{6.77}$$

where ${}^{C}\mathscr{D}^{\alpha}_{\tau}$ denotes the Caputo fractional derivative of order α and $f : [0,1] \times X \to X$ is a continuous function.

The nonlinear fractional differential Equation 6.77 can be written as

$$\begin{aligned} \text{u(t)} &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau - \\ &\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau + \\ &\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^{\sigma} \left[\int_0^{\tau} (\tau - z)^{\alpha-1} f(z, u(z))dz \right] d\tau. \end{aligned} \tag{6.78}$$

A function $u \in C(I, X)$ is a solution of the fractional differential integral equation (6.78) if and only if u is a solution of the nonlinear fractional differential equation (6.77).

We define a quasi partial S_b metric on X as

$$S_{qp_b}(u,u,v) = \left(\sup_{t \in [0,1]} |u(t) - v(t)| \right)^2 + \left(\sup_{t \in [0,1]} |u(t) - v(t)| \right)^2.$$

Then, (X, S_{qp_b}) is a complete quasi-partial S_b metric space.

Now, we prove the following theorem.

Theorem 6.8 *Suppose the following hypotheses hold:*

(i) *there exists* $f \in C(I \times X, X)$ *a continuous in the first and fifth variables;*

(ii) *there exists a continuous function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}_+$*, such that*

$$|f(t,u(\tau) - f(t,v(\tau)| \le 2\vartheta|u(\tau) - v(\tau)|^2,$$

for all $t \in [0,1]$ *and for all* $u,v \in X$ *and a constant* $\vartheta \in \left[0, \frac{1}{s}\right)$ *such that*

$$\vartheta = \left[\frac{t^{\alpha}(2-\sigma^2)(\alpha+1) + 2t(\alpha+\sigma^{(\alpha+1)}+1)}{(2-\sigma^2)\Gamma(\alpha)(\alpha(\alpha+1))}\right]^2.$$

Then, the fractional differential Equation 6.77 *has a common solution as a fixed point* $u^* \in C(I,X)$.

Proof: Let us define $\mathscr{S}, \mathscr{A} : C([0,1]) \to C([0,1])$, with $\mathscr{A} \in \mathscr{S}$ by

$$\begin{aligned} \mathscr{S}\mathrm{u(t)} &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,u(\tau))d\tau - \\ &\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,u(\tau))d\tau + \\ &\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^{\sigma}\left[\int_0^{\tau}(\tau-z)^{\alpha-1}f(z,u(z))dz\right]d\tau, \end{aligned} \tag{6.79}$$

for $t \in [0,1]$, then $\mathscr{S}$ is continuous at the first and fith variables. Suppose that

$$\mathscr{A}u(t) = \int_0^{\tau}(\tau-z)^{\alpha-1}f(z,u(z))dz,$$

this implies that $\mathscr{A} \in \mathscr{S}$ and $\mathscr{A}$ posses a fixed point $u^* \in \mathscr{S}$. To prove the existence of fixed point of η, we prove that η is continuous in the first and fifth variables of the implicit function F_S and is a contraction. To show this, let $\mathscr{S}u \neq \mathscr{A}v$, for all $u,v \in [0,1]$. By the hypothesis of Theorem 6.8, we have

$$\begin{aligned} |\mathscr{A}\mathrm{u} - \mathscr{S}\mathrm{v}| &= 2\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,u(\tau))d\tau - \right. \\ &\quad \frac{2t}{(2-\nu^2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,u(\tau))d\tau + \\ &\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^{\sigma}\left[\int_0^{\tau}(\tau-z)^{\alpha-1}f(z,u(z))dz\right]d\tau \end{aligned}$$

$$\begin{aligned}
&-\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, v(s))ds + \\
&\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau, v(\tau))d\tau - \\
&\left.\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^\sigma \left[\int_0^\tau (\tau-z)^{\alpha-1} f(z, v(z))dz\right] d\tau\right|^2,
\end{aligned}$$

$$\begin{aligned}
\leq\ & 2\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} |f(\tau, u(\tau)) - f(\tau, v(\tau))| d\tau + \\
&\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} |f(\tau, u(\tau)) - f(\tau, v(\tau))| d\tau, + \\
&\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^\sigma \left[\int_0^\tau (\tau-z)^{\alpha-1} |f(z, u(z)) - f(z, v(z))| dz\right] d\tau\Bigg)^2, \\
\leq\ & 2\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} |u(\tau) - v(\tau)| d\tau + \\
&\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1} |u(\tau) - v(\tau)| d\tau + \\
&\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^\sigma \left[\int_0^\tau (\tau-z)^{\alpha-1} |u(z) - v(z)| dz\right] d\tau\Bigg)^2,
\end{aligned}$$

$$\begin{aligned}
=\ & 2\Bigg(\frac{1}{\Gamma(\alpha)}\|u-v\|_\infty \int_0^t (t-\tau)^{\alpha-1} d\tau + \\
&\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\|u-v\|_\infty \int_0^1 (1-\tau)^{\alpha-1} d\tau + \\
&\frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\|u-v\|_\infty \int_0^\sigma \left[\int_0^\tau (\tau-z)^{\alpha-1} dz\right] d\tau\Bigg)^2,
\end{aligned}$$

$$\begin{aligned}
\leq\ & \left[\frac{t^\alpha}{\alpha\Gamma(\alpha)} + \frac{2t}{(2-\sigma^2)\alpha\Gamma(\alpha)} + \frac{2t\sigma^{\alpha+1}}{(2-\sigma^2)\alpha(\alpha+1)\Gamma(\alpha)}\right]^2 2\|u-v\|_\infty^2, \\
\leq\ & 2\vartheta\|u-v\|_\infty^2. \qquad (6.80)
\end{aligned}$$

This implies that

$$\|\mathscr{A}u - \mathscr{S}v\|_\infty \ \leq\ 2\vartheta\|u-v\|_\infty^2.$$

Thus, for each $u, v \in X$, we have

$$S_{qp_b}(\mathscr{A}u, \mathscr{A}u, \mathscr{S}v) \ \leq\ \vartheta S_{qp_b}(u, u, v). \qquad (6.81)$$

For $\vartheta \in [0, \frac{1}{s})$ and the condition $((FS1) - (FS2))$ shows that $\mathscr{A}$-$\mathscr{S}$ is a contraction mapping on X, since all the hypotheses of Theorem (6.8) are satisfied. Therefore, there exists $u^* \in C(I)$ a common fixed point of $\mathscr{A}$ and $\mathscr{S}$, that is, u^* is a solution to fractional nonlinear differential equation (6.77).

CHAPTER SEVEN

CONCLUSION, RECOMENDATION AND FUTURE WORK

7.1 Introduction

This study aimed at developing several fixed point theorems for contractive mappings in abstract spaces with some applications. This was done by extending, generalising and improving existing theorems in literature and adjusting them to apply to the context of the study. The proofs of the theorems were done using rules of logical deduction, mathematical induction and, when appropriate, proof by contradiction using the Banach contraction principle concept. In this chapter, we briefly state the conclusions from this study, provide a few recommendations, future works, limitations of the study and a list of published papers.

7.2 Conclusion

We conclude from the study that it is possible to extend some fixed point theorems proved for mappings in metric spaces into another class of abstract spaces. Fixed point theorem refers to an equation $x = Tx$; usually, the theorem gives conditions for the existence and uniqueness of a solution. For example, the function T may be thought of as a mapping. Then a solution x is a point that a mapping leaves fixed.

This study showed several ways to construct, extend, formulate, prove, and generalise fixed point theorems in abstract spaces, first, by using various maps, i.e., single valued map, multivalued map, hybrid map and implicit maps, where the proof is done by finding a coincidence point or common fixed point results.

Also, the generalisation is done by considering relatively large classes of abstract spaces; Cone metric spaces, b-metric spaces, partial b-metric spaces, metric-like spaces, partial metric spaces, quasi partial S_b-metric-like spaces, and G-metric spaces.

Next, the formulation of a fixed point can be done by considering the conditions that ensure a common fixed point: commutativity, containment of ranges of mappings, continuity or weaker conditions. Using $(E.A)$- property, $(JCLR)$- property, (CLR)-property and several others to obtain a fixed point.

Furthermore, the domains of the mappings needed to be complete or at least 0-complete and have non-empty boundaries. We found out that the extended theorems generally retained their form when modified to cater for abstract spaces.

In this work, we have tackled some types of optimal control problems in the presence of the newly proposed integral differential equations, fractional differential equations and ordinary differential equations. In order to obtain the fixed point results, we exploited the techniques mentioned in several books and the fractional integration techniques. As a result, the formulation showed, and the obtained results are analogous when the classical principles are used; but with slight differences. Therefore, we believe in the need of tackling such operators and that this work may initiate the interest of researchers in them as they can also be used in modelling some problems considered in various fields of sciences. Therefore, by using all generalisation methods, we conclude that it is possible to obtain new results.

The following are significant contributions of this study to fixed point theory:

The coincidence fixed point for a pair of mappings result given in Theorem 3.4, Theorem 3.7, Theorem 3.8 and Theorem 3.9. These theorems provide the coincidence fixed point conditions for a substantial class of F-Kannan mappings on various abstract spaces. We proved a coincidence fixed point theorem for a pair of F-Kannan mapping in a generalised metric space. Furthermore, a fixed point theorem is proved for F-Kannan-Suzuki type mappings in TVS valued cone metric space. Specifically, we introduced, extended and generalised the results due to Batra *et al.* (2020) and Morandi and Alimohammadi (2011). In addition, some examples were provided to validate the results and an application to integral equations, nonlinear fractional differential equations and ordinary differential equations for damped forced oscillation, which generalises several well-known results in the literature.

The fixed point for multivalued result given in Theorem 4.5, Theorem 4.7 and Theorem 4.9. These theorems provide the fixed point conditions for a substantial class of Hady-Rogers contraction mappings on various abstract spaces. We proved a fixed point theorem for multi-valued mapping using α-F-contraction in partial metric spaces. Furthermore, a fixed point theorem is proved for F-Hardy-Rogers multi-valued mappings in ordered partial metric spaces. Specifically, this paper is motivated by the works by Ali and Kamran (2016), Sgroi and Vetro (2013). We also provided illustrative examples and an application to integral equations, which generalises some well-known results in the literature. These results have some applications in several areas of applied mathematics, especially in the Volterra type integral equation.

The coincidence result is given in Theorem 5.4. This theorem provides the coincidence conditions for a substantial class of non-self mappings on various abstract spaces. The study is motivated by the results obtained by Aserkar and Gandhi (2020) in metric space. We proved a fixed point theorem for common fixed point for two hybrid pairs of coincidentally idempotent non-self mappings in weakly partial b-metric space, which satisfies joint common limit range property in a generalised (F, ξ, η)-contraction, these results have some applications in many areas of applied mathematics, especially in hybrid differential equations.

We also proved the results for p-hybrid mappings via common limit range (CLR) property in G-metric space setting. The theorems proved here are Theorem 5.5 and Theorem 5.6, that generalise the results due to Karapinar *et al.* (2013), Karapinar (2009), Nashine *et al.* (2018), Karapinar *et al.* (2019) and Karapinar *et al.* (2020) from metric space and quasi partial metric notion to G-metric space. We give an example to demonstrate the results.

7.3 Recommendations

Mathematicians can extend this study to develop theorems that state the conditions for the existence of fixed points for non-self mappings, non-expansive mappings in abstract spaces.

The abstract spaces have found an application in nonlinear analysis. However, more research needs to be done to identify applications in other areas of mathematics, chemistry, physics. Finally, it would be interesting to investigate the fixed point theory of particular abstract spaces. This means we could move from examining the general abstract space and look into the specific application of abstract spaces.

7.3.1 Limitations of the Study

This study focuses on contraction mappings, which forces the spaces to be continuous and complete for the existence of a fixed point.

7.3.2 Future work

In future work, we intend to investigate the fixed point theorems and its applications in physics to show how X-ray machine works concerning the fixed point theorems and its application to matrix equations.

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APPENDIX

The content of this thesis is based on the following articles

Wangwe L and Kumar S 2021 A Common Fixed Point Theorem for Generalised F-Kannan Mapping in Metric Space with Applications. *Abstract and Applied Analysis,* **2021**(Article ID 6619877): 1-12.https://doi.org/10.1155/2021/6619877 (Chapter 3)

Wangwe L and Kumar S 2021 Some common Fixed-point theorems for a pair of p-hybrid mappings via common limit range property in *G*-metric. *Results in Nonlinear Analysis,* **4**(2021): 87-104. https://doi.org/10.53006/rna.907704 (Chapter 5)

Wangwe L and Kumar S 2021 A Common fixed point theorem for hybrid pair of mappings in a generalised (F,ε,η)-contraction in weak Partial *b*- metric spaces with some Application. *Advances in the Theory of Nonlinear Analysis and its Applications,* **4**(5): 531-550. https://doi.org/10.31197/atnaa.934778 (Chapter 5)

Wangwe L and Kumar S 2021 Fixed Point Theorems for Multi-valued α-F- contractions in Partial metric spaces with Some Application. *Results in Nonlinear Analysis,* **4**(3), (2021): 130-148. https://doi.org/10.53006/rna.937822 (Chapter 4)

Wangwe L and Kumar S 2021 Fixed point theorem for multivalued non-self mappings in partial symmetric spaces, *Topological Algebra and Applications.* **2021**(9): 20-36. https://doi.org/10.1515/taa.2021-0102 (Chapter 4)

Wangwe L and Kumar S 2021 Fixed point theorem for $\mathscr{S}$-Φ-Ψ-contraction mappings in ordered partial metric space with an application to integral equation, *Nonlinear Studies.* **28**(1): 1207-1223. (Chapter 3)

Wangwe L and Kumar S 2021 Common fixed-point theorems under implicit contractive condition using E. A property on metric-like spaces employing with an arbitrary binary relation and an application to integral equation, *International Journal of Nonlinear Analysis and Applications (Iran).* (Chapter 6) [Accepted]

Wangwe L and Kumar S 2021 Common Fixed Point Theorems for Generalised F-Kannan-Suzuki type Mapping in TVS valued Cone Metric Space with Some Applications*Iournal of Mathematics.* (Chapter 3) [Accepted]

The following articles are under review:

Wangwe L and Kumar S 2021 Fixed Point Theorems for Multi-valued Jaggi-Hardy-Rogers type-F-$F_{\mathscr{C}}$-contractions in partially ordered b-metric space with an Application. (Chapter 4) [under review]

Wangwe L and Kumar S 2021 Generalised Common fixed point theorem for weakly compatible mappings via implicit contractive relation in quasi-partial S_b-metric space with some applications (Chapter 6) [under review]