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PERFORMANCE DYNAMICS OF BATCH ARRIVAL FEEDBACK QUEUE WITH GENERAL SECOND OPTIONAL SERVICES, SERVER BREAKDOWN AND STANDBY SERVER

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Abstract. This paper aims to analyze the steady state behavior of a bulk input general service queue with second optional service, breakdowns, general repair, and delay times. The server may experience random failures during the first essential and second optional services, and we assume there is a delay before the server starts the repair process. The system is equipped with a standby server, which provides service to the customers only when the main server is under repair due to sudden failure or during the delay time for repair to start. Moreover, the service times first essential service and second optional service, delay times, and repair times have a general distribution, while the breakdown times and standby service times follow an exponential distribution. The steady state probabilities are computed using the probability generating function. Finally, numerical illustrations of performance measures are provided.

1. Introduction.

Bulk arrival queueing models serve as a fundamental framework for analyzing and designing various real-world systems efficiently. Their applications extend to areas such as telephone exchange operations, perishable-item inventory management, computer and communication networks, data and voice transfer processes, and manufacturing systems. In queueing theory, the representation of batch arrival was presented in the work of Oduol and Ardil [25], where the authors analyzed a single server queue with fixed size batch arrivals under steady state and transient conditions. Similar work on batch arrivals has been studied by Singh et al. [27]. The authors investigated an $M^X/G/1$ queueing model incorporating server vacations and customer balking, in which individuals may choose not to enter the system during both busy and vacation periods, each governed by distinct balking probabilities. Malik et al. [22] analyzed a retrial batch arrival queueing framework characterized by working vacations, an additional optional service, and negative customer arrivals occurring while the server is serving a positive customer.

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Several authors have contributed to the study of batch arrival queue system, including Jeyakumar and Senthilnathan [12], Jailaxmi et al. [11], Vignesh et al. [31], Ayyappan and Nirmala [3, 4], and Lavanya et al. [15].

Recently, queueing models incorporating customer impatience have attracted considerable interest. These models have practical uses in various service sectors and online commerce. For a detailed review of related literature on queueing systems involving customer impatience, readers are referred to Bouchentouf et al. [9], Bouchentouf et al.[8], Laxmi et al. [19] and the references therein.

Batch arrival queueing systems with second optional service ($\mathcal{SOS}$) have gained much attention in the literature. A variety of queueing models with the inclusion of $\mathcal{SOS}$ have been investigated by several authors. Baruah et al. [7] studied group arrival queues with reneging and second elective service. They considered reneging when the server is on a break periods inaccessible during the system breakdown. The study by Uma and Punniyamoorthy [29] investigated a batch arrival system with a secondary optional service. In their model, customers are served in groups of size M with two categories of general service, and arrivals may choose to balk whenever the server is either occupied or on vacation. Recently, Laxmi et al. [17] used the generating function technique to investigate the steady state queue system with group arrival, optional service and customer impatience.

Nevertheless, depending on a number of factors such as the quality of the service, it may be necessary to repeat customer service (feedback) in queueing systems. Baruah et al. [6] applied the concept of re-service and balking in a batch arrival queueing system providing two general heterogeneous services. Bouchentouf et al. [10] analyzed a multi-server queue with customers' impatience and Bernoulli feedback under a variant of multiple vacations. Recently, Laxmi et al. [18] investigated bulk arrival queues that incorporates $\mathcal{SOS}$ along with a feedback mechanism. The model considers that after receiving first essential service ($\mathcal{FES}$), customers can rejoin the line and request the service once more if their needs are not met. Other studies on feedback are found in Ayyappan and Shyamala [5], Vignesh et al. [30], Soundararajan and Josephine [28], Abdollahi and Salehi Rad [1], Mytalas and Zazanis [24] and the reference therein.

In practical situations, customer service can often be disrupted by unexpected server failures. Therefore, the server cannot provide service unless it is repaired. However, in numerous cases, the repair process cannot commence right away because of unavailability of required tools or the absence of a technician. For instance, a manufacturing system might be down due to a broken part, and if the part is not available, the system cannot be repaired until it can be supplied with necessary equipment and resources. Khalaf et al. [13] investigated a batch arrival queueing model that incorporates random server breakdowns and general repair times. Maragathasundari and Sowmiyah [23] investigated an $M/G/1$ queueing framework with general delay and repair distributions, assuming that the server restoration cannot be accomplished in a single phase but instead requires two successive stages to complete. Later, Madan and Malalla [21] presented a batch arrival queue with $\mathcal{SOS}$, breakdowns, delay and repair where the arriving batches are admitted into the system based on a policy of restricted admissibility. Significant contributions addressing server breakdowns and repair mechanisms are reported in Ayyappan and Karpagam [2], Rajan et al. [26], and related studies.

In industrial systems such as manufacturing, power plants and telecommunication systems, the system may fail and the repair process will take more time, which in turn causes considerable losses in production, potential opportunities, goodwill, and revenue. To mitigate these risks, the system often maintains a standby server to ensure the required level of availability of service. When the essential operating server encounters a failure, the standby server is instantly activated and takes over the assigned operations. Khalaf et al. [14] presented the queue system with vacation, breakdown, general repair, standby server under steady state domain. Where the standby server provides the service during periods when the main server on vacation or undergoing repairs. Recently, Laxmi et al. [16] employed the generating function technique to explore the transient and steady state of queue system with batch service, $\mathcal{SOS}$, repairable breakdown and standby server.

Although numerous studies exist, no research has specifically addressed batch arrival queueing models that integrate a standby server along with breakdowns, feedback, delays and repair times. Therefore, for many scenarios that occur in real-time, these queueing models are more useful for analyzing. These models can help in calculating the customer's waiting time, the server's utilization, etc,. Motivated by this, in this paper, we analyze bulk arrival general service queue with $\mathcal{SOS}$, feedback, breakdowns, standby server, general delay and repair times. The aim of this study is to obtain:

- The steady state probabilities and some important system characteristics of the model.
- Impact of different parameters on the system's operations metrics.

The remainder of this paper is organized as follows. Section 2 outlines the system description, while Section 3 derives the governing equations and establishes the equilibrium solution. Various performance metrics are discussed in Section 4 and Section 5 presents some important special cases. Section 6 provides numerical examples supported by tables and graphical illustrations. Finally, Section 7 concludes the study.

2. **The model.** This work investigates an $M^X/G/1$ queueing framework incorporating $\mathcal{SOS}$, feedback, random server breakdowns, as well as general delay and repair distributions. The essential features of the model are outlined below:

- Customers arrive in random batch sizes denoted by $\mathcal{X}$, following a Poisson stream occurring with chance $\mathcal{P}(\mathcal{X} = \mathrm{j}) = \mathrm{c_j}$. The probability that exactly j customers $(\mathrm{j} = 1, 2, \ldots)$ enter the system during a short interval $(\mathrm{t}, \mathrm{t} + \mathrm{dt})$ is given by $\lambda \mathrm{c_j dt}$, where $\lambda > 0$ represents the mean batch arrival rate. Furthermore, $\sum_{\mathrm{j}=1}^{\infty} \mathrm{c_j} = 1$ and $0 \leq \mathrm{c_j} \leq 1$ for all j. The mean batch size is $\mathcal{E}[\mathcal{X}] = \sum_{\mathrm{j}=1}^{\infty} jc_j$.
- The service times for $\mathcal{FES}$ and $\mathcal{SOS}$ are considered to follow an arbitrary probability distribution with functions $\mathcal{F}(\mathrm{y})$ and $\mathcal{H}(\mathrm{y})$ and the density functions are $\mathrm{f(y)}$ and $\mathrm{h(y)}$, respectively. Let $\mu(\mathrm{y})\mathrm{dy}, \beta(\mathrm{y})\mathrm{dy}$ be the conditional probabilities of the completion of $\mathcal{FES}$ and $\mathcal{SOS}$, respectively during the interval $(\mathrm{y}, \mathrm{y} + \mathrm{dy})$ with elapsed service time y, so that

$$\mu(\mathrm{y}) = \frac{\mathrm{f(y)}}{1 - \mathcal{F}(\mathrm{y})} \text{ and } \mathrm{f(y)} = \mu(\mathrm{y})\mathrm{e}^{-\int_0^{\mathrm{y}} \mu(\mathrm{t})\mathrm{dt}},$$

$$\beta(\mathrm{y}) = \frac{\mathrm{h(y)}}{1 - \mathcal{H}(\mathrm{y})} \text{ and } \mathrm{h(y)} = \beta(\mathrm{y})\mathrm{e}^{-\int_0^{\mathrm{y}} \beta(\mathrm{t})\mathrm{dt}}.$$

- Upon finishing $\mathcal{FES}$, a customer proceeds to $\mathcal{SOS}$ with probability r_0, may return to the queue with probability r_2 if unsatisfied, or departs from the system with probability r_1, such that $r_0 + r_1 + r_2 = 1$.
- The system may have random breakdowns during $\mathcal{FES}$ and $\mathcal{SOS}$. We assume that breakdowns in $\mathcal{FES}$ and $\mathcal{SOS}$ follow a Poisson process with mean breakdown rate α.
- After a breakdown, the server's repair does not commence immediately. We assume a repair-initiation delay that follows a general probability distribution with function $\mathcal{J}(y)$ and density $j(y)$. Let $\varphi(y)dy$ denote the conditional probability that this delay is completed within $(y, y+dy)$, given an elapsed delay time of y, so that

$$\varphi(y) = \frac{j(y)}{1-\mathcal{J}(y)} \text{ and } j(y) = \varphi(y)e^{-\int_0^y \varphi(t)dt}.$$

- Furthermore, the repair duration is considered to follow a general probability distribution characterized by the function $\mathcal{G}(y)$ and density function $g(y)$. Define $\phi(y)dy$ as the conditional probability that the repair is completed within the interval $(y, y+dy)$, given that y units of repair time have already elapsed, so that

$$\phi(y) = \frac{g(y)}{1-\mathcal{G}(y)} \text{ and } g(y) = \phi(y)e^{-\int_0^y \phi(t)dt}.$$

- The standby server starts serving the customers as soon as there is a breakdown during $\mathcal{FES}$ or $\mathcal{SOS}$ with service times following an exponential distribution with rate δ.

2.1. **Mathematical model formulation.** The system's state at time t is depicted by the Markov process $\{\mathcal{L}_q(t), \mathcal{Y}(t), \mathcal{X}_1(t), \mathcal{X}_2(t), t \geq 0\}$, where $\mathcal{L}_q(t)$ denotes the number of customers in the queue at time t, $\mathcal{X}_i(t)$ (where $i = 1$ corresponds to $\mathcal{FES}$ and $i = 2$ to $\mathcal{SOS}$) represents the elapsed service time of the batch currently being processed and $\mathcal{Y}(t)$ indicates the server's state at time t, explained by:

$$\mathcal{Y}(t) = \begin{cases} 0, & \text{The system has no customers and the server is idle at time } t, \\ 1, & \mathcal{FES} \text{ is being offered at time } t, \\ 2, & \mathcal{SOS} \text{ is being offered at time } t. \end{cases}$$

The Markov process has the following state space:

$$\chi = \{\{0,0\}U\{n,j,\mathcal{X}_1\}U\{n,j,\mathcal{X}_2\} : n \geq 0, j = 1,2, \mathcal{X}_1 \geq 0, \mathcal{X}_2 \geq 0\}$$

The following expressions denote the probabilities in this model:

- $\mathcal{Q}(t)$ denotes the probability that the queue is empty and the server is idle.
- $\mathcal{P}_{n,1}(y,t)$ denotes the probability that, at time t, the system contains n (≥ 0) customers waiting in the queue, one customer currently being served, and the primary server is functioning under $\mathcal{FES}$, given that the elapsed service time is y.
- $\mathcal{P}_{n,2}(y,t)$ is defined as the probability that, for a given time t, the queue length equals n (≥ 0), exactly one unit is being served, and the server operates under $\mathcal{SOS}$ with an elapsed service time of y.

- $\mathcal{R}_n(y,t)$ represents the probability that at time t, there are $n \geq 0$ customers waiting in the queue, and the system is under repair with elapsed repair time y.
- $\mathcal{D}_n(y,t)$ represents the probability that at time t, there are $n \geq 1$ customers waiting in the queue, the server is inactive due to a breakdown, and the system is in the delay phase, with elapsed delay time y. The delay phase occurs only if at least one customer is present after the breakdown. Hence, the state corresponding to delay with zero customers is impossible, which implies $D_0(y) = 0$ for all $y \geq 0$.

Note that the above probabilities at steady state are denoted by $\mathcal{Q}$, $\mathcal{P}_{n,1}(y)$, $\mathcal{P}_{n,2}(y)$, $\mathcal{R}_n(y)$, and $\mathcal{D}_n(y)$, respectively.

3. **Model's steady state solution.** Based on the formulation outlined in the preceding section, the steady state differential-difference equations can be expressed in the following form:

$$\lambda\mathcal{Q} = r_1\int_0^\infty \mathcal{P}_{0,1}(y)\mu(y)dy + \int_0^\infty \mathcal{P}_{0,2}(y)\beta(y)dy + \int_0^\infty \mathcal{R}_0(y)\phi(y)dy, \tag{1}$$

$$\frac{d}{dy}\mathcal{P}_{0,1}(y) + \Big(\lambda + \mu(y) + \alpha\Big)\mathcal{P}_{0,1}(y) = 0, \tag{2}$$

$$\frac{d}{dy}\mathcal{P}_{n,1}(y) + \Big(\lambda + \mu(y) + \alpha\Big)\mathcal{P}_{n,1}(y) = \lambda\sum_{i=1}^{n} c_i\mathcal{P}_{n-i,1}(y),\ n \geq 1, \tag{3}$$

$$\frac{d}{dy}\mathcal{P}_{0,2}(y) + \Big(\lambda + \beta(y) + \alpha\Big)\mathcal{P}_{0,2}(y) = 0, \tag{4}$$

$$\frac{d}{dy}\mathcal{P}_{n,2}(y) + \Big(\lambda + \beta(y) + \alpha\Big)\mathcal{P}_{n,2}(y) = \lambda\sum_{i=1}^{n} c_i\mathcal{P}_{n-i,2}(y),\ n \geq 1, \tag{5}$$

$$\frac{d}{dy}\mathcal{R}_0(y) + \Big(\lambda + \phi(y) + \delta\Big)\mathcal{R}_0(y) = \delta\mathcal{R}_1(y), \tag{6}$$

$$\frac{d}{dy}\mathcal{R}_n(y) + \Big(\lambda + \phi(y) + \delta\Big)\mathcal{R}_n(y) = \lambda\sum_{i=1}^{n} c_i\mathcal{R}_{n-i}(y) + \delta\mathcal{R}_{n+1}(y)\ n \geq 1, \tag{7}$$

$$\frac{d}{dy}\mathcal{D}_0(y) = 0, \tag{8}$$

$$\frac{d}{dy}\mathcal{D}_n(y) + \Big(\lambda + \varphi(y) + \delta\Big)\mathcal{D}_n(y) = \lambda\sum_{i=1}^{n} c_i\mathcal{D}_{n-i}(y) + \delta\mathcal{D}_{n+1}(y)\ n \geq 1. \tag{9}$$

The equations above are solved using the boundary conditions specified as

$$\mathcal{P}_{n,1}(0) = r_1\int_0^\infty \mathcal{P}_{n+1,1}(y)\mu(y)dy + r_2\int_0^\infty \mathcal{P}_{n,1}(y)\mu(y)dy + \int_0^\infty \mathcal{P}_{n+1,2}(y)\beta(y)dy + \int_0^\infty \mathcal{R}_{n+1}(y)\phi(y)dy + \lambda c_{n+1}\mathcal{Q},\ n \geq 0, \tag{10}$$

$$\mathcal{P}_{n,2}(0) = r_0\int_0^\infty \mathcal{P}_{n,1}(y)\mu(y)dy,\ n \geq 0, \tag{11}$$

$$\mathcal{D}_0(0) = 0, \tag{12}$$

$$\mathcal{D}_n(0) = \alpha \int_0^\infty \mathcal{P}_{n-1,1}(y)dy + \alpha \int_0^\infty \mathcal{P}_{n-1,2}(y)dy, \ n \geq 1, \tag{13}$$

$$\mathcal{R}_n(0) = \int_0^\infty \mathcal{D}_n(y)\varphi(y)dy, \ n \geq 0. \tag{14}$$

The set of equations (1)–(14) are analyzed through the use of probability generating functions ($\mathcal{PGF}s$) , which are described below:

$$\mathcal{P}_i(y,u) = \sum_{n=0}^{\infty} \mathcal{P}_{n,i}(y)u^n, \ |u| \leq 1, \ y > 0, \ i = 1, 2,$$

$$\mathcal{R}(y,u) = \sum_{n=0}^{\infty} \mathcal{R}_n(y)u^n, \ \mathcal{D}(y,u) = \sum_{n=0}^{\infty} \mathcal{D}_n(y)u^n, \ |u| \leq 1, \ y > 0,$$

$$\mathcal{C}(u) = \sum_{i=1}^{\infty} c_i u^i, \ |u| \leq 1., \mathcal{E}[\mathcal{X}] = \sum_{i=1}^{\infty} ic_i, C'(1) = \mathcal{E}[\mathcal{X}].$$

Proposition 3.1. *If* $y > 0$, *then*

$$(I) \ \frac{\partial}{\partial y}\mathcal{P}_1(y,u) + \Big(\lambda(1-\mathcal{C}(u)) + \mu(y) + \alpha\Big)\mathcal{P}_1(y,u) = 0, \tag{15}$$

$$(II) \ \frac{\partial}{\partial y}\mathcal{P}_2(y,u) + \Big(\lambda(1-\mathcal{C}(u)) + \beta(y) + \alpha\Big)\mathcal{P}_2(y,u) = 0, \tag{16}$$

$$(III) \ \frac{\partial}{\partial y}\mathcal{D}(y,u) + \left(\lambda(1-\mathcal{C}(u)) + \varphi(y) + \delta - \frac{\delta}{u}\right)\mathcal{D}(y,u) = 0, \tag{17}$$

$$(IV) \ \frac{\partial}{\partial y}\mathcal{R}(y,u) + \left(\lambda(1-\mathcal{C}(u)) + \phi(y) + \delta - \frac{\delta}{u}\right)\mathcal{R}(y,u) = 0. \tag{18}$$

Proof. (I) Multiplying equation (3) by u^n and sum from $n = 1$ to ∞ and add equation (2), this leads to the result.

(II) In a similar manner, utilizing equations (4) and (5), the required result can be derived.

(III) From equations (6) and (7), the equation (17) is obtained.

(IV) The result follows from equations (8) and (9). □

Proposition 3.2. *If* $y > 0$, *then*

$$\mathcal{P}_1(y,u) = \mathcal{P}_1(0,u)e^{-\eta(u)y - \int_0^y \mu(t)dt}, \tag{19}$$

$$\mathcal{P}_2(y,u) = \mathcal{P}_2(0,u)e^{-\eta(u)y - \int_0^y \beta(t)dt}, \tag{20}$$

$$\mathcal{D}(y,u) = \mathcal{D}(0,u)e^{-\zeta(u)y - \int_0^y \varphi(t)dt}, \tag{21}$$

$$\mathcal{R}(y,u) = \mathcal{R}(0,u)e^{-\zeta(u)y - \int_0^y \phi(t)dt}, \tag{22}$$

where $\eta(u) = \lambda(1-\mathcal{C}(u)) + \alpha$, $\zeta(u) = \lambda(1-\mathcal{C}(u)) + \delta - \frac{\delta}{u}$.

Proof. By performing integration of equations (15), (16), (17) and (18) over the interval $[0, y]$, the required result is obtained. □

Proposition 3.3. *If* $y > 0$, *then*

$$\int_0^\infty \mathcal{P}_1(y,u)\mu(y)dy = \mathcal{P}_1(0,u)\mathcal{F}^*[\eta(u)], \tag{23}$$

$$\int_0^\infty \mathcal{P}_2(y,u)\beta(y)dy = \mathcal{P}_2(0,u)\mathcal{H}^*[\eta(u)], \tag{24}$$

$$\int_0^\infty \mathcal{D}(y,u)\varphi(y)dy = \mathcal{D}(0,u)\mathcal{J}^*[\zeta(u)], \tag{25}$$

$$\int_0^\infty \mathcal{R}(y,u)\phi(y)dy = \mathcal{R}(0,u)\mathcal{G}^*[\zeta(u)], \tag{26}$$

where the Laplace–Stieltjes Transform of $\mathcal{F}(y)$, $\mathcal{H}(y)$, $\mathcal{J}(y)$, $\mathcal{G}(y)$, *are* $\mathcal{F}^*[\eta(u)], \mathcal{H}^*[\eta(u)]$, $\mathcal{J}^*[\zeta(u)]$ *and* $\mathcal{G}^*[\zeta(u)]$, *respectively and are given by*

$$\mathcal{F}^*[\eta(u)] = \int_0^\infty e^{-\eta(u)y}d\mathcal{F}(y),\ \mathcal{H}^*[\eta(u)] = \int_0^\infty e^{-\eta(u)y}d\mathcal{H}(y),$$
$$\mathcal{J}^*[\zeta(u)] = \int_0^\infty e^{-\zeta(u)y}d\mathcal{J}(y),\ \mathcal{G}^*[\zeta(u)] = \int_0^\infty e^{-\zeta(u)y}d\mathcal{G}(y).$$

Proof. Equations (19), (20), (21) and (22) are multiplied by $\mu(y)$, $\beta(y)$, $\varphi(y)$, and $\phi(y)$, respectively, and then integrated with respect to y, yielding the required result. □

Proposition 3.4. *The* $\mathcal{PGF}s$ $\mathcal{P}_i(u)$, $i = 1, 2$, $\mathcal{R}(u)$ *and* $\mathcal{D}(u)$ *are given by*

$$\mathcal{P}_1(u) = \frac{\big[1-\mathcal{F}^*[\eta(u)]\big]\lambda(\mathcal{C}(u)-1)\mathcal{Q}}{\mathcal{Y}(u)}, \tag{27}$$

$$\mathcal{P}_2(u) = \frac{r_0\mathcal{F}^*[\eta(u)]\big[1-\mathcal{H}^*[\eta(u)]\big]\lambda\big(\mathcal{C}(u)-1\big)\mathcal{Q}}{\mathcal{Y}(u)}, \tag{28}$$

$$\mathcal{D}(u) = \frac{\Big[1-\mathcal{F}^*[\eta(u)] + r_0\mathcal{F}^*[\eta(u)]\big(1-\mathcal{H}^*[\eta(u)]\big)\Big]\alpha u\big[1-\mathcal{J}^*[\zeta(u)]\big]\lambda(\mathcal{C}(u)-1)\mathcal{Q}}{\zeta(u)\mathcal{Y}(u)}, \tag{29}$$

$$\mathcal{R}(u) = \frac{\Big[1-(1-r_0)\mathcal{F}^*[\eta(u)](1-\mathcal{H}^*[\eta(u)])\Big]\alpha u\mathcal{J}^*[\zeta(u)]\big[1-\mathcal{G}^*[\zeta(u)]\big]\lambda(\mathcal{C}(u)-1)\mathcal{Q}}{\zeta(u)\mathcal{Y}(u)}, \tag{30}$$

where

$$\mathcal{P}_i(u) = \int_0^\infty \mathcal{P}_i(y,u)dy,\ i=1,2,\ \mathcal{D}(u) = \int_0^\infty \mathcal{D}(y,u)dy,\ \mathcal{R}(u) = \int_0^\infty \mathcal{R}(y,u)dy,$$
$$\mathcal{Y}(u) = \eta(u)\Big[u - r_1\mathcal{F}^*[\eta(u)] - r_2u\mathcal{F}^*[\eta(u)] - r_0\mathcal{F}^*[\eta(u)]\mathcal{H}^*[\eta(u)]\Big]$$
$$-\alpha u\Big[1-\mathcal{F}^*[\eta(u)] + r_0\mathcal{F}^*[\eta(u)] - r_0\mathcal{F}^*[\eta(u)]\mathcal{H}^*[\eta(u)]\Big]\mathcal{J}^*[\zeta(u)]\mathcal{G}^*[\zeta(u)].$$

Proof. Integrating equations (19), (20), (21) and (22) by parts, we obtain

$$\mathcal{P}_1(u) = \mathcal{P}_1(0,u)\left(\frac{1-\mathcal{F}^*[\eta(u)]}{\eta(u)}\right), \tag{31}$$

$$\mathcal{P}_2(u) = \mathcal{P}_2(0,u)\left(\frac{1-\mathcal{H}^*[\eta(u)]}{\eta(u)}\right), \tag{32}$$

$$\mathcal{D}(\mathrm{u}) = \mathcal{D}(0,\mathrm{u})\left(\frac{1-\mathcal{J}^*[\zeta(\mathrm{u})]}{\zeta(\mathrm{u})}\right), \tag{33}$$

$$\mathcal{R}(\mathrm{u}) = \mathcal{R}(0,\mathrm{u})\left(\frac{1-\mathcal{G}^*[\zeta(\mathrm{u})]}{\zeta(\mathrm{u})}\right). \tag{34}$$

Now, we have to find $\mathcal{P}_1(0,\mathrm{u}),\ \mathcal{P}_2(0,\mathrm{u}),\ \mathcal{D}(0,\mathrm{u}),\ \mathcal{R}(0,\mathrm{u})$.

By multiplying equation (10) with u^n, sum from $n = 0$ to infinity, the following expression is obtained:

$$\begin{aligned}\mathrm{u}\mathcal{P}_1(0,\mathrm{u}) = \lambda\mathcal{C}(\mathrm{u})\mathcal{Q} + \mathrm{r}_1\int_0^\infty \mathcal{P}_1(\mathrm{y},\mathrm{u})\mu(\mathrm{y})\mathrm{dy} + \mathrm{ur}_2\int_0^\infty \mathcal{P}_1(\mathrm{y},\mathrm{u})\mu(\mathrm{y})\mathrm{dy}\\ + \int_0^\infty \mathcal{P}_2(\mathrm{y},\mathrm{u})\beta(\mathrm{y})\mathrm{dy} + \int_0^\infty \mathcal{R}(\mathrm{y},\mathrm{u})\phi(\mathrm{y})\mathrm{dy}\\ - \left[\mathrm{r}_1\int_0^\infty \mathcal{P}_{0,1}(\mathrm{y})\mu(\mathrm{y})\mathrm{dy} + \int_0^\infty \mathcal{P}_{0,2}(\mathrm{y})\beta(\mathrm{y})\mathrm{dy} + \int_0^\infty \mathcal{R}(\mathrm{y})\phi(\mathrm{y})\mathrm{dy}\right].\end{aligned} \tag{35}$$

Substituting equations (1), (23), (24) and (26) into equation (35), we have,

$$\begin{aligned}\mathrm{u}\mathcal{P}_1(0,\mathrm{u}) =& \mathrm{r}_1\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{P}_1(0,\mathrm{u}) + \mathrm{r}_2\mathrm{u}\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{P}_1(0,\mathrm{u})\\ &+ \mathcal{P}_2(0,\mathrm{u})\mathcal{H}^*[\eta(\mathrm{u})] + \mathcal{R}(0,\mathrm{u})\mathcal{G}^*[\zeta(\mathrm{u})] + \lambda(\mathcal{C}(\mathrm{u})-1)\mathcal{Q}.\end{aligned} \tag{36}$$

Similarly, from equations (11) and (23), we get

$$\mathcal{P}_2(0,\mathrm{u}) = \mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{P}_1(0,\mathrm{u}), \tag{37}$$

and from equations (12), (13), (19), (20) and (37), we obtain

$$\mathcal{D}(0,\mathrm{u}) = \frac{\alpha\mathrm{u}}{\eta(\mathrm{u})}\Big[\big[1-\mathcal{F}^*[\eta(\mathrm{u})]\big] + \mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\big[1-\mathcal{H}^*[\eta(\mathrm{u})]\big]\Big]\mathcal{P}_1(0,\mathrm{u}). \tag{38}$$

Similarly, from equations (14), (25) and (38), we get

$$\mathcal{R}(0,\mathrm{u}) = \frac{\alpha\mathrm{u}}{\eta(\mathrm{u})}\Big[\big[1-\mathcal{F}^*[\eta(\mathrm{u})]\big] + \mathrm{r}_0\big[1-\mathcal{H}^*[\eta(\mathrm{u})]\big]\mathcal{F}^*[\eta(\mathrm{u})]\Big]\mathcal{J}^*[\zeta(\mathrm{u})]\mathcal{P}_1(0,\mathrm{u}). \tag{39}$$

Now, to find $\mathcal{P}_1(0,\mathrm{u})$, substitute equations (37) and (39) into equation (36), we get

$$\mathcal{P}_1(0,\mathrm{u}) = \frac{\lambda(\mathcal{C}(\mathrm{u})-1)\eta(\mathrm{u})\mathcal{Q}}{\mathcal{Y}(\mathrm{u})}, \tag{40}$$

where $\mathcal{Y}(\mathrm{u}) = \big[\mathrm{u} - \mathrm{r}_1\mathcal{F}^*[\eta(\mathrm{u})] - \mathrm{r}_2\mathrm{u}\mathcal{F}^*[\eta(\mathrm{u})] - \mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{H}^*[\eta(\mathrm{u})]\big]\eta(\mathrm{u}) - \alpha\mathrm{u}\big[1 - \mathcal{F}^*[\eta(\mathrm{u})] + \mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})] - \mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{H}^*[\eta(\mathrm{u})]\big]\mathcal{J}^*[\zeta(\mathrm{u})]\mathcal{G}^*[\zeta(\mathrm{u})]$. To find $\mathcal{P}_2(0,\mathrm{u})$, substitute equation (40) into equation (37), we get

$$\mathcal{P}_2(0,\mathrm{u}) = \frac{\mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\lambda(\mathcal{C}(\mathrm{u})-1)\eta(\mathrm{u})\mathcal{Q}}{\mathcal{Y}(\mathrm{u})}. \tag{41}$$

To find $\mathcal{D}(0,\mathrm{u})$, substitute equation (40) into equation (38), we get

$$\mathcal{D}(0,\mathrm{u}) = \frac{\Big[1-\mathcal{F}^*[\eta(\mathrm{u})] + \mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\big[1-\mathcal{H}^*[\eta(\mathrm{u})]\big]\Big]\alpha\mathrm{u}\lambda(\mathcal{C}(\mathrm{u})-1)\mathcal{Q}}{\mathcal{Y}(\mathrm{u})}. \tag{42}$$

To find $\mathcal{R}(0,\mathrm{u})$, substitute equation (40) into equation (39), we get

$$\mathcal{R}(0,\mathrm{u})=\frac{\alpha\mathrm{u}\big[1-\mathcal{F}^*[\eta(\mathrm{u})]+\mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\big[1-\mathcal{H}^*[\eta(\mathrm{u})]\big]\big]\mathcal{J}^*[\zeta(\mathrm{u})]\lambda(\mathcal{C}(\mathrm{u})-1)\mathcal{Q}}{\mathcal{Y}(\mathrm{u})}. \tag{43}$$

□

After substituting equations (40), (41), (42) and (43) into equations (31), (32), (33) and (34), respectively, we get the result.

Proposition 3.5. *The $\mathcal{PGF}$ corresponding to the queue length can be expressed as*

$$\mathcal{P}_\mathrm{q}(\mathrm{u})=\frac{\mathcal{N}(\mathrm{u})}{\mathcal{M}(\mathrm{u})} \tag{44}$$

where

$$\begin{aligned}\mathcal{N}(\mathrm{u})=&\Big[\lambda\big(\mathcal{C}(\mathrm{u})-1\big)\Big]\Big[1-\mathcal{F}^*[\eta(\mathrm{u})](1-\mathrm{r}_0)-\mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{H}^*[\eta(\mathrm{u})]\Big]\\&\times\Big[\zeta(\mathrm{u})+\alpha\mathrm{u}\big[1-\mathcal{J}^*[\zeta(\mathrm{u})]\mathcal{G}^*[\zeta(\mathrm{u})]\big]\Big]\mathcal{Q},\\\mathcal{M}(\mathrm{u})=&\zeta(\mathrm{u})\mathcal{Y}(\mathrm{u})=\Big(\lambda(1-\mathcal{C}(\mathrm{u}))+\delta-\frac{\delta}{u}\Big)\Big(\big[\mathrm{u}-(\mathrm{r}_1+\mathrm{r}_2)\mathcal{F}^*[\eta(\mathrm{u})]\\&-\mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{H}^*[\eta(\mathrm{u})]\big]\eta(\mathrm{u})-\alpha\mathrm{u}[1-\mathcal{F}^*[\eta(\mathrm{u})]+\mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\\&-\mathrm{r}_0\mathcal{F}^*[\eta(\mathrm{u})]\mathcal{H}^*[\eta(\mathrm{u})]]\mathcal{J}^*[\zeta(\mathrm{u})]\mathcal{G}^*[\zeta(\mathrm{u})]\Big).\end{aligned}$$

Proof. Consider the $\mathcal{PGF}$ of the queue length ($\mathcal{P}_\mathrm{q}(\mathrm{u})$) that is independent of the system state and defined as.

$$\mathcal{P}_\mathrm{q}(\mathrm{u})=\mathcal{P}_1(\mathrm{u})+\mathcal{P}_2(\mathrm{u})+\mathcal{D}(\mathrm{u})+\mathcal{R}(\mathrm{u}). \tag{45}$$

Substituting equations (27), (28), (29) and (30) in equation (45), we get the result. □

Proposition 3.6. *The system has no customers, and the server remains idle with probability characterized by:*

$$\mathcal{Q}=\frac{\Big(-\lambda\mathcal{E}[\mathcal{X}]+\delta\Big)\Big(\alpha[1-\mathrm{r}_2\mathcal{F}^*(\alpha)]-\tilde{\gamma}(\alpha)\big[\lambda\mathcal{E}[\mathcal{X}]+\alpha(1+\mathcal{J}'(0)+\mathcal{G}'(0))\big]\Big)}{\alpha\big(-\lambda\mathcal{E}[\mathcal{X}]+\delta\big)[1-\mathrm{r}_2\mathcal{F}^*(\alpha)]-\alpha\tilde{\gamma}(\alpha)\big[-\lambda\mathcal{E}[\mathcal{X}]+\delta[1+\mathcal{J}'(0)+\mathcal{G}'(0)]\big]}, \tag{46}$$

Proof. To determine $\mathcal{Q}$, the normalizing condition can be applied as

$$\mathcal{P}_\mathrm{q}(1)+\mathcal{Q}=1. \tag{47}$$

It is evident that setting $\mathrm{u}=1$ transforms equation (44) into an indeterminate form $\left(\frac{0}{0}\right)$. Hence, by applying L'Hôpital's rule, we obtain:

$$\mathcal{P}_\mathrm{q}(1)=\frac{\lambda\mathcal{E}[\mathcal{X}]\mathcal{Q}\tilde{\gamma}(\alpha)\Big[-\lambda\mathcal{E}[\mathcal{X}]+\delta-\alpha(\mathcal{J}^{*\prime}(0)+\mathcal{G}^{*\prime}(0))\Big]}{\Big(-\lambda\mathcal{E}[\mathcal{X}]+\delta\Big)\Big(\alpha[1-\mathrm{r}_2\mathcal{F}^*(\alpha)]-\tilde{\gamma}(\alpha)\big[\lambda\mathcal{E}[\mathcal{X}]+\alpha(1+\mathcal{J}'(0)+\mathcal{G}'(0))\big]\Big)}, \tag{48}$$

where $\tilde{\gamma}(\alpha) = 1 - (1 - r_0)\mathcal{F}^*(\alpha) - r_0\mathcal{F}^*(\alpha)\mathcal{H}^*(\alpha), \mathcal{G}^*(0) = 1, \mathcal{C}(1) = 1, \mathcal{J}^*(0) = 1$, and $\mathcal{C}'(1) = \mathcal{E}[\mathcal{X}]$ represents the first moment of the arriving customer batch.

Substituting equation (48) in (47), and performing algebraic calculations, we obtain the result. □

4. **Measures of performance.** In this section, we employ the $\mathcal{PGF}$ of the queue length derived earlier to determine the mean queue size and the waiting time of the customer in the queue.

Let $\mathcal{L}_q$ represent the mean number of customers in the queue, defined as:

$$\mathcal{L}_q = \lim_{u \to 1} \frac{d}{du} \mathcal{P}_q(u), \tag{49}$$

here, $\mathcal{P}_q(u)$ represents the $\mathcal{PGF}$ of the queue length. Evaluating the limit of the differential of $\mathcal{P}_q(u)$ at $u = 1$ converts equation (49) into undefined form $\left(\frac{0}{0}\right)$. Applying appropriate limit rule and computing the differential at $u = 1$, we obtain:

$$\mathcal{L}_q = \frac{\mathcal{N}'''(1)\mathcal{M}''(1) - \mathcal{M}'''(1)\mathcal{N}''(1)}{3(\mathcal{M}'(1))^2}. \tag{50}$$

By performing algebraic manipulations, the second and third derivatives at $u = 1$ can be derived, yielding:

$$\begin{aligned}
\mathcal{N}(u) &= \Big[\lambda(\mathcal{C}(u) - 1)\Big]\Big[1 - \mathcal{F}^*[\eta(u)](1 - r_0) - r_0\mathcal{F}^*[\eta(u)]\mathcal{H}^*[\eta(u)]\Big] \\
&\quad \times \Big[\zeta(u) + \alpha u\big[1 - \mathcal{J}^*[\zeta(u)]\mathcal{G}^*[\zeta(u)]\big]\Big]\mathcal{Q}, \\
\mathcal{N}''(1) &= 2\lambda\mathcal{E}[\mathcal{X}]\mathcal{Q}\big[1 - \mathcal{F}^*[\alpha](1 - r_0) - r_0\mathcal{F}^*[\alpha]\mathcal{H}^*[\alpha]\big] \\
&\quad \times \big[-\lambda\mathcal{E}[\mathcal{X}] + \delta - \alpha(\mathcal{J}^{*\prime}(0) + \mathcal{G}^{*\prime}(0))\big], \\
\mathcal{N}'''(1) &= 3\mathcal{Q}\lambda\mathcal{E}[\mathcal{X}]\Big(2\left[-\lambda\mathcal{E}[\mathcal{X}] + \delta - \alpha\mathcal{J}^{*\prime}(0) - \alpha\mathcal{G}^{*\prime}(0)\right] \\
&\quad \times \big[-\mathcal{F}^{*\prime}(\alpha) + r_0\mathcal{F}^{*\prime}(\alpha) - r_0\mathcal{F}^{*\prime}(\alpha)\mathcal{H}^*(\alpha) - r_0\mathcal{F}^*(\alpha)\mathcal{H}^{*\prime}(\alpha)\big] \\
&\quad + \big[1 - \mathcal{F}^*(\alpha) + r_0\mathcal{F}^*(\alpha) - r_0\mathcal{F}^*(\alpha)\mathcal{H}^*(\alpha)\big]\big[-\lambda\mathcal{E}[\mathcal{X}(\mathcal{X} - 1)] \\
&\quad - 2\delta - 2\alpha\mathcal{G}^{*\prime}(0) - 2\alpha\mathcal{J}^{*\prime}(0) - 2\alpha\mathcal{J}^{*\prime}(0)\mathcal{G}^{*\prime}(0) - \alpha\mathcal{J}^{*\prime\prime}(0) - \alpha\mathcal{G}^{*\prime\prime}(0)\big]\Big) \\
&\quad + 3\mathcal{Q}\lambda\mathcal{E}[\mathcal{X}(\mathcal{X} - 1)]\big[1 - \mathcal{F}^*(\alpha) + r_0\mathcal{F}^*(\alpha) - r_0\mathcal{F}^*(\alpha)\mathcal{H}^*(\alpha)\big]\big[-\lambda\mathcal{E}[\mathcal{X}] \\
&\quad + \delta - \alpha(\mathcal{J}^{*\prime}(0) + \mathcal{G}^{*\prime}(0))\big], \\
\mathcal{M}(u) &= \Big(\lambda(1 - \mathcal{C}(u)) + \delta - \frac{\delta}{u}\Big)\Big(\big[u - (r_1 + r_2)\mathcal{F}^*[\eta(u)] - r_0\mathcal{F}^*[\eta(u)]\mathcal{H}^*[\eta(u)]\big] \\
&\quad \times \eta(u) - \alpha u[1 - (1 - r_0)\mathcal{F}^*[\eta(u)] - r_0\mathcal{F}^*[\eta(u)]\mathcal{H}^*[\eta(u)]]\mathcal{J}^*[\zeta(u)]\mathcal{G}^*[\zeta(u)]\Big), \\
\mathcal{M}''(1) &= 2\Big(-\lambda\mathcal{E}[\mathcal{X}] + \delta\Big)\Big(\alpha[1 - r_2\mathcal{F}^*(\alpha)] - \big[1 - \mathcal{F}^*[\alpha](1 - r_0)
\end{aligned}$$

$$- r_0\mathcal{F}^*[\alpha]\mathcal{H}^*[\alpha]\big]\big[\lambda\mathcal{E}[\mathcal{X}] + \alpha(1 + \mathcal{J}'(0) + \mathcal{G}'(0))\big]\Big),$$

$$\mathcal{M}'''(1) = 3\Big(- \lambda\mathcal{E}[\mathcal{X}(\mathcal{X}-1)] - 2\delta\Big)\Big(\alpha(1 - r_2 * F(\alpha)) - [1 - \mathcal{F}^*(\alpha) + r_0\mathcal{F}^*(\alpha)$$
$$- r_0\mathcal{F}^*(\alpha)\mathcal{H}^*(\alpha)]\big[\lambda\mathcal{E}[\mathcal{X}] + \alpha + \alpha(\mathcal{J}^{*\prime}(0) + \mathcal{G}^{*\prime}(0))\big]\Big) + 3$$
$$\times \Big(- \lambda\mathcal{E}[\mathcal{X}] + \delta\Big)\Big(\big[\lambda\mathcal{E}[\mathcal{X}(\mathcal{X}-1)] + 2\alpha\mathcal{J}^{*\prime}(0) + 2\alpha\mathcal{G}^{*\prime}(0)$$
$$+ \alpha\mathcal{J}^{*\prime\prime}(0) + \alpha\mathcal{G}^{*\prime\prime}(0)\big][1 - \mathcal{F}^*(\alpha) + r_0\mathcal{F}^*(\alpha) - r_0\mathcal{F}^*(\alpha)\mathcal{H}^*(\alpha)]$$
$$+ 2\alpha\mathcal{J}^{*\prime}(0)\mathcal{G}^{*\prime}(0) - 2\lambda\mathcal{E}[\mathcal{X}]\big[1 - r_2\mathcal{F}^*(\alpha)\big] - 2r_2\alpha\mathcal{F}^{*\prime}(\alpha)$$
$$- 2\big[\lambda\mathcal{E}[\mathcal{X}] + \alpha(1 + \mathcal{J}^{*\prime}(0) + \mathcal{G}^{*\prime}(0))\big]$$
$$\times [-\mathcal{F}^{*\prime}(\alpha) + r_0\mathcal{F}^{*\prime}(\alpha) - r_0\mathcal{F}^{*\prime}(\alpha)\mathcal{H}^*(\alpha) - r_0\mathcal{F}^*(\alpha)\mathcal{H}^{*\prime}(\alpha)]\Big),$$

where, $\mathcal{C}''(1) = \mathcal{E}[\mathcal{X}^2 - \mathcal{X}] = \mathcal{E}[\mathcal{X}(\mathcal{X}-1)]$ represents the second moment of the arriving batch. By substituting $\mathcal{N}''$, $\mathcal{N}'''$, $\mathcal{M}''$, $\mathcal{M}'''$ into (50), the closed-form expression for $\mathcal{L}_q$ is obtained.

Let $\mathcal{W}_q$ denote the mean queueing delay. Applying Little's formula gives

$$\mathcal{W}_q = \frac{\mathcal{L}_q}{\lambda\mathcal{E}[\mathcal{X}]},$$

where $\mathcal{L}_q$ is found in equation (50).

5. **Special cases.** This section discusses particular cases that arise as direct consequence of the main findings of this study.

(i) Assuming that there are no breakdowns ($\alpha = 0$) and no standby server ($\delta = 0$), equation (44) reduces to

$$\mathcal{P}_q(u) = \frac{-\mathcal{Q}\Big[(1 - \mathcal{F}^*[\lambda - \lambda\mathcal{C}(u)]) + r_0\mathcal{F}^*[\lambda - \lambda\mathcal{C}(u)](1 - \mathcal{H}^*[\lambda - \lambda\mathcal{C}(u)])\Big]}{u - r_1\mathcal{F}^*[\lambda - \lambda\mathcal{C}(u)] - r_2 u F^*[\lambda - \lambda\mathcal{C}(u)] - r_0\mathcal{F}^*[\lambda - \lambda\mathcal{C}(u)]\mathcal{H}^*[\lambda - \lambda\mathcal{C}(u)]}.$$

Since $\mathcal{F}^*[0] = 1$ and $\mathcal{H}^*[0] = 1$, $P_q(1)$ is undefined at $u = 1$; therefore, we use L'Hôpital's rule to evaluate it. Using the normalizing condition, the idle probability $\mathcal{Q}$ is obtained as

$$\mathcal{Q} = \frac{1 - r_2 + \lambda C'(1)\left[\mathcal{F}^{*'}(0) + r_0\mathcal{H}^{*'}(0)\right]}{1 - r_2}.$$

The obtained results are consistent with the $M^X/G/1$ queue with feedback and $\mathcal{SOS}$ model studied in Laxmi et al. [18] (equations (39) and (40)) when the balking parameter $b = 1$ in their work. For the parameter values $\lambda = 2$, $\mu = 5$, and $\beta = 4$, the numerical results coincide as shown in Table 5.

(ii) We assume no feedback ($r_2 = 0$), no $\mathcal{SOS}$ ($r_0 = 0$) and no standby server ($\delta = 0$) the model reduces to an $M^X/G/1$ queueing system with server breakdowns and general delay and repair distributions. We observe that this result coincides with the outcome of an $M^X/G/1$ queue with Bernoulli schedule, random breakdowns, general vacation, delay, and repair times as a special case when general vacation time is taken to be zero in their work Khalaf et al. [13]. For the parameter values $\alpha = 0.2, \mu = 6$, and $\beta = 5$, the numerical results coincide as shown in Table 6.

(iii) Assuming that there are no breakdowns ($\alpha = 0$), no standby server ($\delta = 0$), no feedback ($r_2 = 0$), and that the probability of joining $\mathcal{SOS}$ is given by $r_0 = r$, and $r_1 = 1 - r$, with no bulk arrival, equation (44) reduces to

$$\mathcal{P}_{\mathrm{q}}(\mathrm{u}) = \frac{-\mathcal{Q}\Big[1 - \mathcal{F}^*[\lambda - \lambda \mathrm{u}] + \mathrm{r}\mathcal{F}^*[\lambda - \lambda \mathrm{u}](1 - \mathcal{H}^*[\lambda - \lambda \mathrm{u}])\Big]}{\mathrm{u} - (1 - \mathrm{r})\mathcal{F}^*[\lambda - \lambda \mathrm{u}] - \mathrm{r}\mathcal{F}^*[\lambda - \lambda \mathrm{u}]\mathcal{H}^*[\lambda - \lambda \mathrm{u}]}.$$

Since $P_q(1)$ is undefined at $\mathrm{u} = 1$, we utilize L'Hôpital's rule, assuming that $-\mathcal{F}^{*'}(0) = 1/\mu_1$ and $-\mathcal{H}^{*'}(0) = 1/\mu_2$. We get

$$\mathcal{P}_{\mathrm{q}}(1) = \frac{\mathcal{Q}\left(r\lambda + \lambda\mu_2/\mu_1\right)}{r\lambda + (\mu_2 - \lambda\mu_2/\mu_1)}.$$

Using the normalizing condition, the idle probability $\mathcal{Q}$ is obtained as

$$\mathcal{Q} = \frac{\mu_2 - \lambda\mu_2/\mu_1 - r\lambda}{\mu_2}.$$

Therefore, the utilization factor of the system is given as

$$\rho = 1 - \mathcal{Q} = \frac{\lambda}{\mu_1} + \frac{r\lambda}{\mu_2}.$$

The obtained results are consistent with the $M/G/1$ queue with $\mathcal{SOS}$ model studied in Madan [20] (equations (35), (36), and (37)). For the parameter values $\mu = 6, \beta = 5$, and $r = 0.2$ the numerical results coincide as shown in Table 7.

6. Numerical investigation.

This section provides numerical demonstrations of the previously obtained results and analyzes the impact of system parameters on the corresponding performance measures. The primary aim is to highlight the impact of the variables ($\lambda,\ \mu,\ \beta,\ \rho,\ \mathrm{r}_0,\ \mathrm{r}_1,\ \mathrm{r}_2$) on $\mathcal{L}_{\mathrm{q}}$, $\mathcal{W}_{\mathrm{q}}$ and $\mathcal{Q}$. The busy probability, $\rho = 1 - \mathcal{Q}$.

TABLE 1. The influence of r_2 and r_0 on $\mathcal{L}_{\mathrm{q}}$, $\mathcal{W}_{\mathrm{q}}$ and $\mathcal{Q}$

r_2	r_0	$\mathcal{Q}$	ρ	$\mathcal{L}_{\mathrm{q}}$	$\mathcal{W}_{\mathrm{q}}$
0.1	0.5	0.320603	0.679397	1.81570	0.907852
0.2	0.4	0.296830	0.703170	2.06395	1.031970
0.3	0.3	0.267606	0.732394	2.43553	1.217760
0.4	0.2	0.230812	0.769188	3.04801	1.524000
0.5	0.1	0.183071	0.816929	4.23243	2.116210

Table 1 illustrates the effect of the feedback probability r_2 and the joining probability r_0 for $\mathcal{SOS}$ on the idle server probability $\mathcal{Q}$, the expected queue length $\mathcal{L}_{\mathrm{q}}$, and the expected waiting time in the queue $\mathcal{W}_{\mathrm{q}}$. In this case, the service times under $\mathcal{FES}$ and $\mathcal{SOS}$, as well as the delay and repair times, are assumed to follow an exponential distribution. Further, we consider $\lambda = 2,\ \mu = 8,\ \beta = 7,\ \phi = 6,\ \varphi =$

$5,\ \alpha = 2,\ \delta = 1$. For the numerical analysis, we have assumed that $\mathcal{E}[\mathcal{X}] = 1$, and $\mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0$ for the purpose of simplifying the lengthy analytical expressions involving these quantities. From the results, it can be seen that when $r_2 < r_0$, the probability of an empty queue is higher, while both the expected queue length and the waiting time in the queue are smaller compared to the case $r_2 \geq r_1$. This reveals that a higher feedback probability of customers results in a larger queue size because it leads to increased customer waiting times.

TABLE 2. The impact of δ on $\mathcal{L}_q$, ρ, $\mathcal{W}_q$ and $\mathcal{Q}$

δ	$\mathcal{Q}$	ρ	$\mathcal{L}_q$	$\mathcal{W}_q$
0	0.288557	0.711443	2.58687	1.29343
0.5	0.338346	0.661654	1.82737	0.913683
1.0	0.381622	0.618378	1.36939	0.684696
1.5	0.419584	0.580416	1.07412	0.537059

In Table 2, we present the impact of the δ on the $\mathcal{Q}$, ρ, $\mathcal{L}_q$ and $\mathcal{W}_q$. In this context, the service time, delay time, and repair times to follow the exponential distribution and $\lambda = 2,\ \mu = 8,\ \beta = 7, \phi = 6,\ \varphi = 5,\ \alpha = 2,\ r_0 = 0.25,\ r_2 = 0.20,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0, \mathcal{E}[\mathcal{X}] = 1$. As δ increases, the server's idle time $\mathcal{Q}$ increases while the mean queue size $\mathcal{L}_q$ and the mean waiting time of the customers $\mathcal{W}_q$ decrease. All the trends shown by the table synchronize with the expected results.

The effect of λ on $\mathcal{L}_q$ and $\mathcal{Q}$ is presented in Table 3. We investigate the effect of λ in the following three cases. (i) exponential, (ii) Hyper-exponential, (iii) Erlang -κ. It is evident from the table that in all three cases, as λ increases, the idle server probability decreases and the mean length of the queue increases, as one would expect. In exponential case, we take $\mu = 8,\ \beta = 7, \phi = 6,\ \varphi = 5,\ r_0 = 0.25,\ r_2 = 0.20,\ \alpha = 2,\ \delta = 1,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0, \mathcal{E}[\mathcal{X}] = 1$. In Erlang-$\kappa$ distribution case we take $\kappa = 2,\ \mu = 8,\ \beta = 7, \phi = 6,\ \varphi = 5,\ r_0 = 0.25,\ r_2 = 0.20,\ \alpha = 2,\ \delta = 1,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0, \mathcal{E}[\mathcal{X}] = 1$. For numerical analysis, we assume that the service time follows a two-phase hyper-exponential distribution defined by $f(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t},\ t > 0$, where p is the probability of choosing phase. Thus, we take $p = 0.5,\ \mu_1 = 9,\ \mu_2 = 8,\ \beta_1 = 8,\ \beta_2 = 7, \phi_1 = 7, \phi_2 = 6, \varphi_1 = 6,\ \varphi_2 = 5,\ r_0 = 0.25,\ r_2 = 0.20,\ \alpha = 2,\ \delta = 1,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0, \mathcal{E}[\mathcal{X}] = 1$.

TABLE 3. Impact of λ on $\mathcal{L}_q$ and $\mathcal{Q}$ when the service, repair and delay duration follow general distribution

	Exponential		*Hyper − exponential*		*Erlang − κ*	
λ	$\mathcal{Q}$	$\mathcal{L}_q$	$\mathcal{Q}$	$\mathcal{L}_q$	$\mathcal{Q}$	$\mathcal{L}_q$
2	0.381622	1.36939	0.427145	1.078340	0.346753	1.50098
2.2	0.319784	1.90453	0.369860	1.45068	0.281429	2.14643
2.4	0.257946	2.72034	0.312575	1.97688	0.216104	3.20864
2.6	0.196108	4.08234	0.255289	2.76051	0.150779	5.22928
2.8	0.13427	6.74518	0.198004	4.02501	0.0854548	10.4064
3.0	0.0724324	14.0405	0.140718	6.35766	0.0201302	49.4689

The influence of the standby server rate δ and breakdown rate α on $\mathcal{L}_q$ and $\mathcal{Q}$ is illustrated in Table 4. For a fixed breakdown rate α, as δ increases, it leads to an increase in idle time $\mathcal{Q}$ and a decrease in mean queue size $\mathcal{L}_q$. Moreover, for fixed standby rates δ, opposite effects are observed for $\mathcal{Q}$ and $\mathcal{L}_q$ with an increase in α, as expected. The service times of $\mathcal{FES}$ and $\mathcal{SOS}$, as well as the delay time and repair time, are assumed to follow exponential distributions, with $\mu = 8$, $\lambda = 2$, $\beta = 7$, $\varphi = 6$, $\phi = 6$, r0 = 0.25, r2 = 0.25, $\mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0$, and $\mathcal{E}[\mathcal{X}] = 1$.

Table 4. The influence of repair δ and mean breakdown α on $\mathcal{L}_q$, $\mathcal{W}_q$ and $\mathcal{Q}$

δ	α	$\mathcal{Q}$	ρ	$\mathcal{L}_q$	$\mathcal{W}_q$
	$\alpha = 1$	0.52381	0.47619	0.516234	0.258117
$\delta = 3$	$\boldsymbol{\alpha = 2}$	**0.490741**	**0.509259**	**0.691955**	**0.345977**
	$\alpha = 3$	0.466667	0.533333	0.834921	0.41746
	$\alpha = 1$	0.550562	0.449438	0.452419	0.22621
$\delta = 4$	$\boldsymbol{\alpha = 2}$	**0.537815**	**0.462185**	**0.557248**	**0.278624**
	$\alpha = 3$	0.529412	0.470588	0.631808	0.315904
	$\alpha = 1$	0.574468	0.425532	0.416273	0.208136
$\delta = 5$	$\boldsymbol{\alpha = 2}$	**0.576923**	**0.423077**	**0.482949**	**0.241474**
	$\alpha = 3$	0.578947	0.421053	0.524721	0.26236

Table 5 illustrates the effect of the feedback probability r_2 and the joining probability r_0 for $\mathcal{SOS}$ on the idle server probability $\mathcal{Q}$, and the expected queue length $\mathcal{L}_q$. This reveals that a higher feedback probability of customers results in a larger queue size because it leads to increased customer waiting times.

Table 5. Influence of r_2 and r_0 on $\mathcal{Q}$ and $\mathcal{L}_q$ for the special case (i)

The current model results				Results of Laxmi et al. [18] when $b = 1$			
r_2	r_0	$\mathcal{Q}$	$\mathcal{L}_q$	r_2	r_0	$\mathcal{Q}$	$\mathcal{L}_q$
0.1	0.5	0.277778	1.65556	0.1	0.5	0.277778	1.65556
0.2	0.4	0.250000	2.00000	0.2	0.4	0.250000	2.00000
0.3	0.3	0.214286	2.59524	0.3	0.3	0.214286	2.59524
0.4	0.2	0.166667	3.83333	0.4	0.2	0.166667	3.83333
0.5	0.1	0.100000	7.70000	0.5	0.1	0.100000	7.70000

Table 6. Influence of λ on $\mathcal{Q}$ and $\mathcal{L}_q$ for the special case (ii)

The current model results			Results of Khalaf et al. [13]		
λ	$\mathcal{Q}$	$\mathcal{L}_q$	λ	$\mathcal{Q}$	$\mathcal{L}_q$
1	0.873333	1.57756	1	0.873333	1.57756
2	0.826667	7.50462	2	0.826667	7.50462
3	0.860000	15.5437	3	0.860000	15.5437
4	0.973333	19.6805	4	0.973333	19.6805

TABLE 7. Influence of λ on $\mathcal{Q}$ and $\mathcal{L}_q$ for the special case (iii)

The current model results			Results of Madan [20]		
λ	$\mathcal{Q}$	$\mathcal{L}_q$	λ	$\mathcal{Q}$	$\mathcal{L}_q$
1	0.793333	0.0535014	1	0.793333	0.0535014
2	0.586667	0.2893940	2	0.586667	0.2893940
3	0.380000	1.0052600	3	0.380000	1.0052600
4	0.173333	3.9179500	4	0.173333	3.9179500

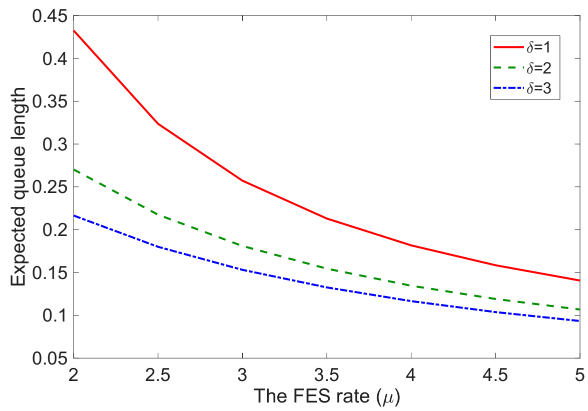

FIGURE 1. The influence of μ on expected queue length for various δ values

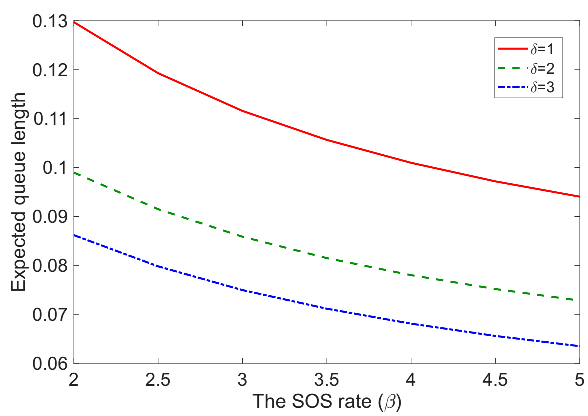

FIGURE 2. The influence of β on expected queue length for various δ values

In Figures 1 to 4, we demonstrate the influence of $\mathcal{FES}$, $\mathcal{SOS}$ service rates, repair time and delay time rate on $\mathcal{L}_q$ for different standby server rate δ. It is noted that $\mathcal{L}_q$ declines as the rate of $\mathcal{FES}$, the $\mathcal{SOS}$ rate, the repair time and the delay time rate increases. Further, we notice that as δ decreases, $\mathcal{L}_q$ grows, which consequently indicates the intuitive results. At this stage, we assume that $\mathcal{FES}$, $\mathcal{SOS}$ repair time and delay times to are assumed to obey an exponential distribution and $r_0 = 0.25, \lambda = 0.5,\ r_2 = 0.20,\ \beta = 7,\ \phi = 6, \varphi = 5, \alpha = 2,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0, \mathcal{E}[\mathcal{X}] = 1$, in Figure 1; $\lambda = 0.5,\ r_0 = 0.25,\ r_2 = 0.20,\ \mu = 8,\ \phi = 6,\ \varphi = 5,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0,\ \alpha = 2,\ \mathcal{E}[\mathcal{X}] = 1$, in Figure 2. Further, we take $\lambda = 0.5,\ r_0 = 0.25,\ r_2 = 0.20,\ \mu = 8,\ \beta = 7,\ \varphi = 5, \alpha = 2,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0,\ \mathcal{E}[\mathcal{X}] = 1$ in Figure 3, and $\lambda = 0.5,\ r_0 = 0.25,\ r_2 = 0.20,\ \mu = 8,\ \beta = 7, \alpha = 2,\ \mathcal{E}[\mathcal{X}] = 1,\ \phi = 6,\ \mathcal{E}[\mathcal{X}(\mathcal{X}-1)] = 0$ in Figure 4.

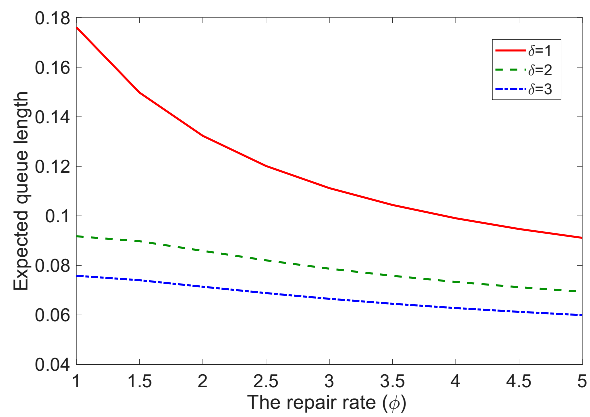

FIGURE 3. The influence of ϕ on $\mathcal{L}_q$ in various standby server rate δ

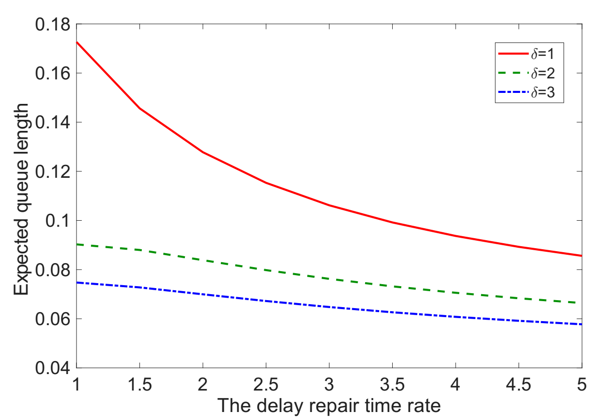

FIGURE 4. The influence of delay repair time(φ) on expected queue length for various δ values

7. **Conclusions.** This study analyzes the steady state behavior of a single-server batch arrival queue with feedback, breakdowns, a standby server, general repair and delay times, and optional service. The supplementary variable method has been used obtain the generating function of probabilities for the system's customer count. An estimation of the mean queue length and customer waiting time was

carried out, followed by a study of special cases. Finally, the numerical findings were depicted using graphs and tables. For future research, the proposed queueing model can be extended in several directions, such as incorporating retrial behavior, adopting a bulk service policy, and considering multi-server aspects.

Conflict of interest. There are no conflicts of interest related to this work.

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