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Finite Difference and Shooting Methods for Two-Point Boundary Value Problems: A Comparative Analysis

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Abstract

Finite difference and shooting methods are popular with numerical practitioners in solving two-point boundary value problems governed by ordinary differential equations. However, the available literature is silent on which method is the most suitable with respect to accuracy, efficiency, stability, and convergence. In this article, finite difference and shooting methods are applied to solve numerically three types of two-point boundary value problems. One problem is governed by a linear non-stiff differential equation, a second problem is governed by a linear stiff differential equation, and a third problem is governed by a non-linear differential equation. The analytical solution of each problem is given. These solutions are used in assessing the accuracy attained by each of the numerical methods. It is known a priori that finite difference schemes based on central difference quotients in approximating derivative terms are numerically stable. Numerical experiments carried out on the three problems lead to the conclusion that, for linear non-stiff equations, linear shooting gives significantly more accurate results compared to finite difference methods. However, in the case of linear stiff equations, the finite difference method gives very accurate results while the shooting method fails totally, displaying serious instability. As for problems governed by nonlinear equations, although both finite difference and shooting methods converge, the shooting method converges significantly faster than its counterpart. With respect to the attribute of efficiency, finite difference methods are more efficient than shooting methods due to the ease with which the Runge - Kutta initial value problem solver can be applied.

Keywords: Two-point boundary value problem, initial value problem, finite difference method, shooting method

1.0 Introduction

Mathematical formulation of a large number of physical phenomena in science and engineering involve single or coupled systems of first or second order ordinary differential equations (ODEs) (Jaiswal and Khandelwal, 2008). Problems modelled by first order ODEs are of *initial value* type. Problems described by second order ODEs are governed by an ODE of the general form

$$y^{\prime\prime} = f(x, y, y^{\prime}), \quad a \le x \le b. \tag{1.1}$$

Problems governed by equations of the form (1.1) are either initial value problems (IVP) or *boundary value* problems (BVP), depending on the number of points at which conditions are imposed on the solution (Burden and Faires, 2011). An initial Value Problem is one in which both conditions are imposed at one point as given in equation (1.2):

$$y^{\prime\prime} = f(x, y, y^{\prime}); \quad y(a) = \alpha, \quad y^{\prime}(a) = \beta. \tag{1.2}$$

A two-point boundary value problem (BVP) is one in which the two conditions are imposed at two different points (Dawkins, 2007), as given in equation (1.5).

$$y^{\prime\prime} = f(x, y, y^{\prime}); \quad y(a) = \alpha, \quad y(b) = \beta \tag{1.3}$$

1.1 Linear Stiff and Non-Stiff Second Order Equations

A linear second order ODE has the general form

$$y^{\prime\prime} + p(x)y^{\prime} + q(x)y + r(x); \quad y(a) = \alpha, \quad y(b) = \beta, \tag{1.4}$$

where $p(x)$, $q(x)$ and $r(x)$ are given functions. Linear ODEs are relatively easy to solve, and a general solution is relatively easy to find. .The general solution of a linear second order equation is a linear combination of two complementary solutions. If the equation is non-linear then, except for some very few special cases, there are no general procedures for finding solutions (Higgins, 2009). The most frequent cases of boundary value problems are those governed by linear second order ordinary differential equations with boundary conditions imposed at two points. In such cases one speaks of a two point boundary value problem (Dawkins, 2007).

A linear second order ODE of the type given by equation (1.4) is said to be stiff if its two complementary solutions behave very differently with either increasing or decreasing values of the independent variable. A typical example of a stiff equation is

$$y^{\prime\prime} - 100y = 0; \quad y(0) = 1, \quad y^{\prime}(0) = -10$$

whose complementary solutions are $y_1(x) = e^{-10x}$, $y_2(x) = e^{10x}$ and hence has the general solution $y(x) = C_1 e^{-10x} + C_2 e^{10x}$ where C_1 and C_2 are arbitrary constants.

2.0 Finite Difference and Shooting Methods

2.1 Finite Difference Method

Two point BVP governed by linear second order ODEs are solved numerically using either finite difference methods or by shooting methods. According to Tu et al. (2008), finite difference methods were first used by Euler in 1768, at the time in when all numerical calculations were done using pen and paper. A finite difference method for solving a two-point boundary value problem given by the set of equations

$$y^{\prime\prime} = f(x, y, y^{\prime}), \quad a \le x \le b; \quad y(a) = \alpha, \quad y(b) = \beta \quad . \tag{2.1}$$

Starts by introducing a mesh $x_i = x_0 + ih, \quad i = 0,1,2,\cdots,n$ on the interval $[a,b]$, where $h = \dfrac{b-a}{n}$, followed by replacing the derivative terms in the differential equation with finite difference quotients relative to the mesh. The exact equation

$$\left.\frac{d^2y}{dx^2}\right|_{x_i} = f\left(x_i, y(x_i), \left.\frac{dy}{dx}\right|_{x_i}\right), \qquad i = 1,2,3,\cdots,n-1 \ . \tag{2.2}$$

is replaced a by difference equation obtained by approximating the derivative terms $\left.\frac{dy}{dx}\right|_{x_i}$ and $\left.\frac{d^2y}{dx^2}\right|_{x_i}$ with the central difference quotients

$$\left.\frac{dy}{dx}\right|_{x_i} = \frac{y_{i+1} - y_{i-1}}{2h}; \quad \left.\frac{d^2y}{dx^2}\right|_{x_i} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} . \tag{2.3}$$

Substituting these difference quotients into (2.2) leads to the system of equations

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f_i\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right), \qquad i = 1,2,3,\cdots,n-1 . \tag{2.4}$$

where $y_0 = \alpha, \quad y_n = \beta$.

Equations (2.4) constitute a system of $n-1$ equations in the $n-1$ unknown values $y(x_i)$ of the solution at the $n-1$ interior mesh points $x_i, \quad i = 1,2,3,\cdots,n-1$ (Morton and Meyers, 2005).

For a linear two-point boundary value problem of the general form

$$y^{//} + p(x)y^{/} + q(x)y = r(x), \quad a < x, < b; \quad y(a) = \alpha, \quad y(b) = \beta \tag{2.5}$$

equation (2.2) becomes

$$\left.\frac{d^2y}{dx^2}\right|_{x_i} + p(x_i)\left.\frac{dy}{dx}\right|_{x_i} + q(x_i)y(x_i) = r(x_i) . \tag{2.6}$$

Again, substitution of the two difference quotients (2.3) into (2.6) leads to the set of linear equations

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + p_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + q_i y_i = r_i; \qquad i = 1,2,\cdots,n-1$$

where $p_i = p(x_i), \quad q_i = q(x_i), \quad r_i = r(x_i)$, with an $O(h^2)$ local truncation error .

Multiplication of each term by h^2 and collecting like terms leads to the difference equation

$$\left(1 - \frac{h}{2}p_i\right)y_{i-1} + \left(-2 + h^2 q_i\right)y_i + \left(1 + \frac{h}{2}p_i\right)y_{i+1} = h^2 r_i . \tag{2.7}$$

Evaluating (2.7) at each of the $n-1$ points x_i, $i = 1,2,...,n-1$ and taking note of the prescribed boundary values, finally leads to the linear system of equations

$$
\begin{array}{rcl}
\left(-2+h^2 q_1\right)y_1+\left(1+\frac{h}{2}p_1\right)y_2 & = & h^2 r_1-\left(1-\frac{h}{2}p_1\right)\alpha := b_1 \\
\left(1-\frac{h}{2}p_2\right)y_1+\left(-2+h^2 q_2\right)y_2+\left(1+\frac{h}{2}p_2\right)y_3 & = & h^2 r_2 := b_2 \\
\left(1-\frac{h}{2}p_3\right)y_2+\left(-2+h^2 q_3\right)y_3+\left(1+\frac{h}{2}p_3\right)y_4 & = & h^2 r_3 := b_3 \\
\vdots & \vdots & \vdots \\
\left(1-\frac{h}{2}p_{n-2}\right)y_{n-3}+\left(-2+h^2 q_{n-2}\right)y_{n-2}+\left(1+\frac{h}{2}p_{n-2}\right)y_{n-1} & = & h^2 r_{n-2} := b_{n-2} \\
\left(1-\frac{h}{2}p_{n-1}\right)y_{n-2}+\left(-2+h^2 q_{n-1}\right)y_{n-1} & = & h^2 r_{n-1}-\left(1-\frac{h}{2}p_{n-1}\right)\beta := b_{n-1}
\end{array}
$$

This system can be expressed in the matrix/vector form $M\underline{Y}=\underline{b}$, where

$$
M=\begin{pmatrix}
\beta_1 & \gamma_1 & & & & & & 0 \\
\alpha_2 & \beta_2 & \gamma_2 & & & & & \\
 & \alpha_3 & \beta_3 & \gamma_3 & & & & \\
 & & \alpha_4 & \beta_4 & \gamma_4 & & & \\
 & & & \ddots & \ddots & \ddots & & \\
 & & & & \alpha_{n-3} & \beta_{n-3} & \gamma_{n-3} & \\
 & & & & & \alpha_{n-2} & \beta_{n-2} & \gamma_{n-1} \\
0 & & & & & & \alpha_{n-1} & \beta_{n-1}
\end{pmatrix},\quad
\underline{Y}=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \end{pmatrix},\quad
\underline{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{n-3} \\ b_{n-2} \\ b_{n-1} \end{pmatrix}. \qquad (2.8)
$$

where $\alpha_i=\left(1-\frac{h}{2}p_i\right)y_{i-1}$; $\beta_i=\left(-2+h^2 q_i\right)y_i$; $\gamma_i=\left(1+\frac{h}{2}p_i\right)y_{i+1}$.

The coefficient matrix M is a triple-diagonal matrix; $\underline{Y}$ is the vector of unknown solution at the interior grid points and $\underline{b}$ is the right hand side vector which contains the known boundary values and values of the function $r(x)$ at the interior node points.

For the general nonlinear case given in (2.4), the system in (2.8) will also be nonlinear but the characteristic triple-diagonal form of the coefficient matrix is maintained. The nonlinear system has the general form

$$
\begin{array}{rcl}
f(y_0,y_1,y_2) & = & 0 \\
f(y_1,y_2,y_3) & = & 0 \\
\vdots & \vdots & \vdots \\
f(y_{n-3},y_{n-2},y_{n-1}) & = & 0 \\
f(y_{n-2},y_{n-1},y_n) & = & 0
\end{array} \qquad (2.9)
$$

Such a system is solved iteratively using either the fixed point method, Newton's method, or a gradient method such as the method of steepest descent. Newton's method is often preferred because of its quadratic rate of convergence.

2.2 Shooting Method

The second most widely used numerical method for solving two point BVPs is the shooting method. The shooting method converts a two point boundary value problem into a pair of initial value problems (Störe and Bulirsch, 2002) and the latter are then solved using any suitable initial value problem solver. However, the most widely used initial value solver is the 4th order Runge-Kutta method (Zhang, 2012). Thus, the general two-point boundary value problem

$y'' = f(x, y, y'); \quad y(a) = \alpha, \quad y(b) = \beta$

is recast as an initial value problem

$$y'' = f(x, y, y'); \quad y(a) = \alpha, \quad y'(a) = s \tag{2.10}$$

where s is an intelligent guess at the slope of the solution at the initial point $x = a$. Figuratively, this approach requires one to find the ***trajectory*** (curve) of a bullet (solution) fired from a gun placed at the point $A(a, \alpha)$, with an angle of inclination the (slope) $y^{/}(a) = s$ and which hits the target at point $B(b, \beta)$.

The shooting method has two versions, linear shooting and nonlinear shooting, depending on whether the governing equation is linear or nonlinear.

2.2.1 Linear Shooting

The shooting method converts the linear two point boundary value problem

$$y^{//}(x) = p(x)y^{/}(x) + q(x)y(x) + r(x); a \le x \le b, \; y(a) = \alpha \;, \; y(b) = \beta \tag{2.11}$$

into a pair of initial value problems

$$y_1^{//}(x) = p(x)y_1^{/}(x) + q(x)y_1(x) + r(x), a \le x \le b, \; y_1(a) = \alpha, \; y_1^{/}(a) = 0 \tag{2.12}$$

$$y_2^{//}(x) = p(x)y_2^{/}(x) + q(x)y_2(x) \quad, a \le x \le b, \; y_2(a) = 0 \;, y_2^{/}(a) = 1 \tag{2.13}$$

such that the solution $y(x)$ of the original problem (2.11) is a linear combination of the form

$y(x) = y_1(x) + \lambda y_2(x)$

where $y_1(x)$ and $y_2(x)$ are the solutions of problems (2.12) and (2.13), respectively. The value of the parameter λ is found by requiring that the solution $y(x)$ satisfies the second boundary condition:

$$y(b) = y_1(b) + \lambda y_2(b) = \beta. \tag{2.14}$$

This leads to the value

$$\lambda = \frac{\beta - y_1(b)}{y_2(b)} \text{, provided that } y_2(b) \ne 0. \tag{2.15}$$

Substituting this value of λ into (2.14) one finally gets the formula

$$y(x_i) = y_1(x_i) + \left[\frac{\beta - y_1(b)}{y_2(b)}\right] y_2(x_i); \qquad i = 1, 2, \cdots, n-1$$

for approximating the solution of the original linear problem (2.11) at the discrete interior points x_i of the partitioned interval $[a, b]$. The values $y_1(x_i)$ and $y_2(x_i)$ at are obtained using the 4th order Runge - Kutta method. It can be shown that the condition $y_2(b) \ne 0$ imposed on the solution $y_2(x)$ is guaranteed by the existence and uniqueness theorem for the solution of a boundary value problem.

2.2.2 Nonlinear Shooting

The shooting method for approximating the solution of the general nonlinear two point boundary value problem

$y^{//} = f(x, y, y^{/}), \quad y(a) = \alpha, \quad y(b) = \beta$

solves the initial value problem

$y^{//} = f(x, y, y^{/}), \quad y(a) = \alpha, \quad y^{/}(a) = s$

using the 4th order Runge-Kutta, where s is a guessed initial slope. The aim is to guess at a slope s such that $y(b) = \beta$. Since it is not possible to guess the true value of s, the initial value problem is solved iteratively with respect to s. Since the secant method is usually used to approximate the root of the function $F(s) = y(b, s) - \beta = 0$, two initial guesses s_1 and s_2 are needed.

The general formula for the iteration is:

$$y_k'' = f(x, y_k, y_k'), \quad y_k(a) = \alpha, \quad y_k'(a) = s_k, \quad k = 1,2,3... \tag{2.16}$$

where s_k is the current guessed initial slope and y_k is the corresponding solution. One starts with a guess s_1, solves the IVP (2.16) to get the value $y_1(b, s_1)$ at $x = b$. The accuracy of the guess s_1 is measured by how close $y_1(b, s_1)$ is to β. Then one guesses at another slope s_2 and solves problem (2.16) to get $y_2(b, s_2)$. Again one compares $y_2(b, s_2)$ with β. From the two end values $y_1(b, s_1)$ and $y_2(b, s_2)$ one iterates to generate a sequence $\{s_k\}$ of slopes and a corresponding sequence $\{y_k(x, s_k\}$ of solutions.

If $\varepsilon > 0$ is a small but acceptable error margin, the stopping criterion for the iteration process is $|y_k(b, s_K) - \beta| < \varepsilon$, together with a preset bound for the maximum number of allowable iterations.

3.0 Numerical Experiments

3.1 Solving Two-Point Boundary Value Problems Governed by Linear Non-Stiff Equations

The two point boundary value problem are considered to be governed by the linear non-stiff differential system

$$y'' = -y; \quad y(0) = 0, \quad y\left(\frac{3\pi}{2}\right) = -1, \tag{3.1}$$

whose analytical solution is $y = \sin(x)$.

Both the finite difference method described in Section 2.2.1 and the shooting method described in Section 2.3.1 are applied in solving the problem; first using an interval of length $h = \frac{3\pi}{10}$ and then halving the interval to get $h = 3\pi/10$. The results are given in Tables 3.1(a) and 3.1(b) and Figures 3.1(a) and 3.1(b).

Table 3.1(a)

Analytical solution and numerical solutions of the linear non-stiff equation (3.1) using finite difference and shooting methods with $h = 3\pi/10$

x	Analytical Solution	Numerical Solutions			
		Finite Difference Method		Shooting Method	
		Solution	Absolute Error	Solution	Absolute Error
0.000000	0.000000	0.000000	*0.000000*	0.000000	*0.000000*
0.942478	0.809017	0.847256	*0.038239*	0.809042	*0.000025*
1.884956	0.951057	0.941925	*0.009132*	0.951071	*0.000014*
2.827433	0.309017	0.199916	*0.109101*	0.309004	*0.000012*
3.769911	-0.587785	-0.719671	*0.131886*	-0.587810	*0.000024*
4.712388	-1.000000	-1.000000	*0.000000*	-1.000000	*0.000000*

Figure 3.1(a)

Graphs of the analytical solution and numerical solutions of the linear non-stiff equation (3.1) using finite difference and shooting methods with $h = 3\pi/10$

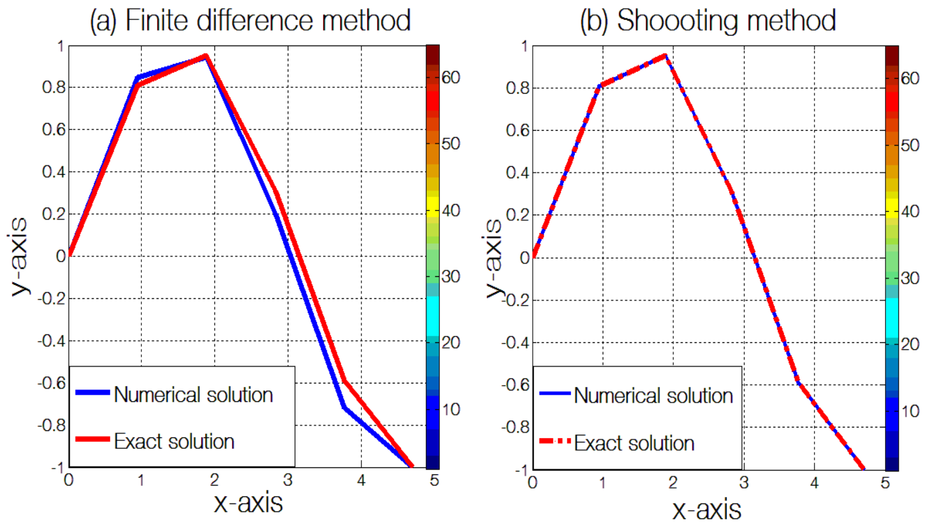
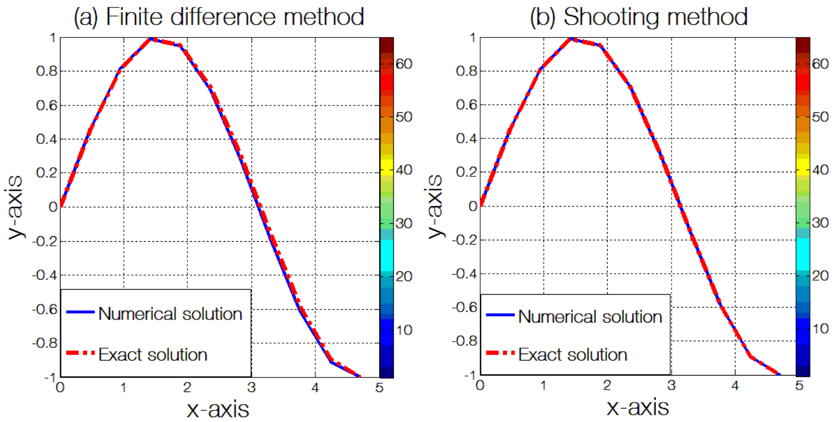


Table 3.1(b)
Analytical solution and numerical solutions of the linear non-stiff equation (3.1) using finite difference and shooting methods with $h = 3\pi/20$

x	Analytical solution	Numerical solutions			
		Finite difference Method		Shooting Method	
		Solution	Absolute Error	Solution	Absolute Error
0.000000	0.000000	0.000000	*0.000000*	0.000000	*0.000000*
0.471238	0.453990	0.458430	*0.004439*	0.454008	*0.000018*
0.942478	0.809017	0.815058	*0.006041*	0.809042	*0.000025*
1.413716	0.987688	0.990689	*0.003001*	0.987712	*0.000024*
1.884956	0.951057	0.946322	*0.004734*	0.951071	*0.000014*
2.356194	0.707106	0.691809	*0.015298*	0.707108	*0.000001*
2.827433	0.309017	0.283669	*0.025348*	0.309004	*0.000001*
3.298672	-0.156434	-0.187465	*0.031030*	-0.156456	*0.000022*
3.769911	-0.587785	-0.616969	*0.029184*	-0.587810	*0.000025*
4.241150	-0.891006	-0.909465	*0.018458*	-0.891026	*0.000019*
4.712388	-1.000000	-1.000000	*0.000000*	-1.000000	*0.000000*

Figure 3.1(b)

Graphs of the analytical solution and numerical solutions of the linear non-stiff equation (3.1) using finite difference and shooting methods with $h = 3\pi/20$



3.2 Solving Stiff, Linear, Two-Point Boundary Value Problems

Consider the stiff, linear, two point boundary vale problem

$$y'' - 100y = 0; \quad y(0) = 1, \quad y(1) = e^{-10} = 0.0000045 \tag{3.2}$$

whose analytical solution is $y = e^{-10x}$.

This problem is solved using the finite difference method and the shooting method with $h = 0.05$. The numerical results are displayed in Table 3.2 and Figure 3.2.

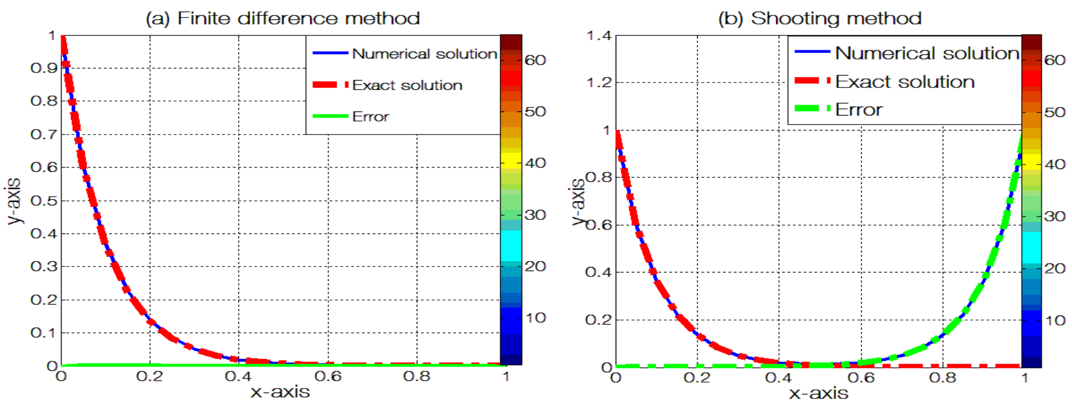
Table 3.2

Analytical and numerical solutions of the linear stiff equation (3.2) using finite difference and shooting methods with $h = 0.05$

x	Analytical Solution	Numerical Solutions			
		Finite Difference Method		Shooting Method	
		Solution	Absolute Error	Solution	Absolute Error
0.000000	1.000000	1.000000	*0*	1.000000	0
0.050000	0.606531	0.609612	*0.003081*	0.606545	*0.000014*
0.100000	0.367879	0.371627	*0.003748*	0.367998	*0.000119*
0.150000	0.223130	0.226548	*0.003418*	0.223160	*0.000030*
0.200000	0.135335	0.138106	*0.002771*	0.135743	*0.000408*
0.250000	0.082085	0.084191	*0.002106*	0.082541	*0.000456*
0.300000	0.049787	0.051324	*0.001537*	0.050787	*0.001000*
0.350000	0.030197	0.031288	*0.001091*	0.031702	*0.001505*
0.400000	0.018316	0.019073	*0.000757*	0.020849	*0.002533*
0.450000	0.011109	0.011627	*0.000518*	0.015209	*0.004100*
0.500000	0.006738	0.007088	*0.000350*	0.013504	*0.006766*
0.550000	0.004087	0.004320	*0.000233*	0.015205	*0.011118*
0.600000	0.002479	0.002633	*0.000154*	0.020806	*0.018327*
0.650000	0.001503	0.001604	*0.000101*	0.031704	*0.030201*
0.700000	0.000912	0.000976	*0.000064*	0.050700	*0.049788*
0.750000	0.000553	0.000593	*0.000040*	0.082635	*0.082082*
0.800000	0.000335	0.000357	*0.000022*	0.1356630	*0.135328*
0.850000	0.000203	0.000210	*0.000007*	0.223324	*0.223121*
0.900000	0.000123	0.000117	*0.000008*	0.367982	*0.367859*
0.950000	7.49e-05	5.18e-05	*0.000023*	0.606591	*0.606516*
1.000000	4.54e-05	4.54e-05	*0*	1.000000	*0.999955*

Figure 3.2

Graphs of the analytical solution and numerical solutions using the finite difference method and the shooting method with $h = 0.05$



3.3 Solving Nonlinear Two-Point Boundary Value Problems

Consider the two point boundary value problem governed by the nonlinear differential system

$$y^{//} = 2y^3 - 6y - 2x^3; \quad y(1) = 2, \quad y(2) = 2.5$$

whose analytical solution is $y = x + \frac{1}{x}$.

Both the finite difference and the shooting methods were applied on the problem using an interval of length $h = 0.05$. Two iterations were performed with each method. The numerical results are given in Table 3.3 and Figures 3.3 (a) and 3.3 (b).

Table 3.3

Analytical and numerical solutions of the nonlinear equation in 2 iterations

x	Analytical Solution	Numerical Solutions			
		Finite Difference Method		Shooting Method	
		Solution	Absolute Error	Solution	Absolute Error
0.000000	2.000000	2.000000	*0.000000*	2.000000	*0.000000*
0.050000	2.002381	2.002403	*0.000022*	2.002381	*0.000000*
0.100000	2.009091	2.009127	*0.000036*	2.009091	*0.000000*
0.150000	2.019565	2.019609	*0.000044*	2.019565	*0.000000*
0.200000	2.033333	2.033381	*0.000048*	2.033333	*0.000000*
0.250000	2.050000	2.050048	*0.000048*	2.050000	*0.000000*
0.300000	2.069231	2.069277	*0.000046*	2.069231	*0.000000*
0.350000	2.090741	2.090785	*0.000044*	2.090741	*0.000000*
0.400000	2.114286	2.114326	*0.000040*	2.114286	*0.000000*
0.450000	2.139655	2.139692	*0.000037*	2.139655	*0.000000*
0.500000	2.166667	2.166700	*0.000044*	2.166667	*0.000000*
0.550000	2.195161	2.195191	*0.000030*	2.195162	*0.000001*
0.600000	2.225000	2.225025	*0.000025*	2.225001	*0.000001*
0.650000	2.256061	2.256082	*0.000021*	2.256061	*0.000000*
0.700000	2.288235	2.288254	*0.000019*	2.288236	*0.000001*
0.750000	2.321429	2.321444	*0.000015*	2.321430	*0.000001*
0.800000	2.355556	2.355568	*0.000012*	2.355557	*0.000001*
0.850000	2.390541	2.390550	*0.000009*	2.390542	*0.000001*

0.900000	2.426316	2.426322	*0.000006*	2.426318	*0.000002*
0.950000	2.462821	2.462824	*0.000003*	2.462824	*0.000001*
1.000000	2.500000	2.500000	*0.000000*	2.500004	*0.000004*

Figure 3.3(a)
Graphs of analytical solution and numerical solution using the Finite Difference Method

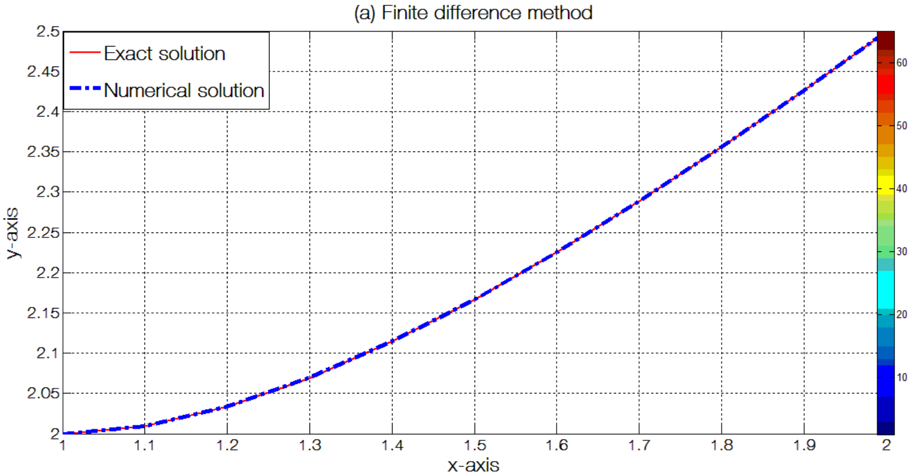
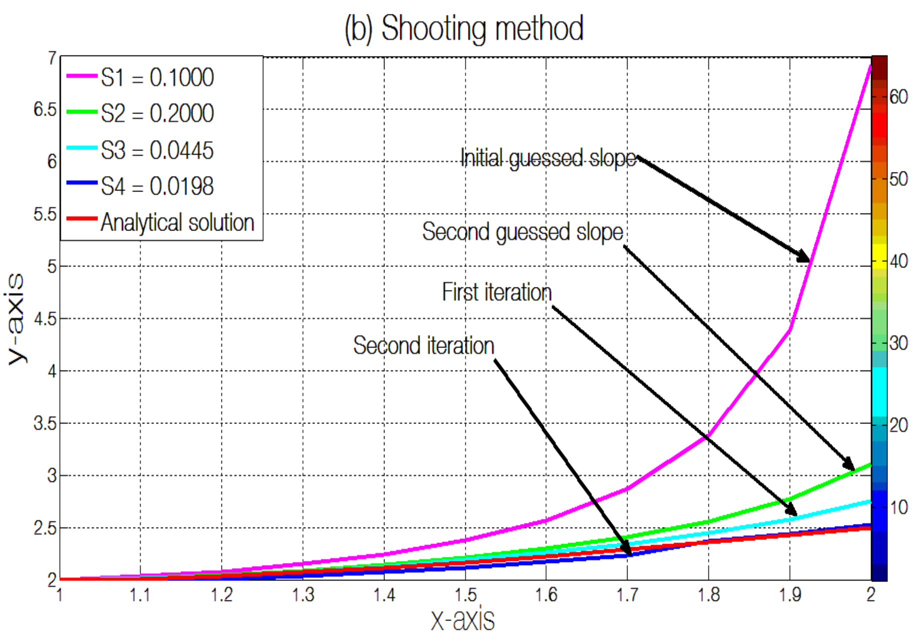


Figure 3.3(b)
Graphs of analytical solution and numerical solution using the Shooting Method



4.0 Comparative Analysis of the Two Methods

Analysis of the two numerical methods in solving numerically a two point BVP governed by a linear non-stiff equation is based on the entries in the absolute error columns in Tables 3.1(a) and 3.1(b) as well as on the pictorial representation of the numerical solutions in Figure 3.1(a) and 3.1(b), which clearly underpin the supremacy of the shooting method over its finite difference counterpart. As expected, mesh refinement through halving the interval length significantly increases the accuracy of the results for each method. However, the increased accuracy is obtained at the cost of doing more computational work and thereby reducing the computational efficiency of each method.

In solving a two point BVP governed by a linear stiff equation, the situation is very different. From the results in Table 3.2(a) and the corresponding graph in 3.2(b) it is quite obvious that the shooting method is not suitable for solving such a problem. However, the finite difference method performs perfectly under such circumstances and is therefore preferred. The only snag here is lack of mechanism to detect a priori that a given governing equation is stiff, especially when the coefficients of the derivative terms are not constants.

Both the shooting method and the finite difference method are quite robust in solving two point BVP governed by a nonlinear differential equation. This is vividly brought out by the results given in Table 3.3 which show that the absolute error in both cases are comparable, although the shooting method seems to be marginally better suited than the finite difference method. Figures 3.3(a) and 3.3(b) underscore the remarks just made.

5.0 Conclusion

The shooting method is more accurate and efficient in approximating solutions of two-point boundary value problems governed by linear and nonlinear ordinary differential equations compared to finite difference methods except where the equation is a stiff equation, in which case finite difference methods are quite robust while the shooting is seriously lacking. The observations are summed up in tabular form as follows:

Type of equation		Finite difference	Shooting
Linear	Non-stiff	• Less accurate • Efficient • Numerically stable	• More accurate • Inefficient • Numerically stable
	Stiff	• Accurate • Efficient • Stable	• Inaccurate • Less efficient • Unstable
Non-linear		• Requires 2 iterations • More efficient • Less accurate • Converges	• Requires 2 iterations • Less efficient • More accurate • Converges

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