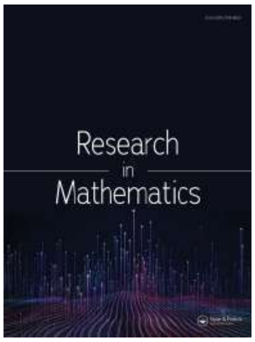

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RESEARCH ARTICLE

Fixed point theorems for Rus-Hicks-Rhoades contractive mappings in orthogonal quasi-metric spaces with applications to orthogonal system models

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ABSTRACT

In this study, we explore fixed point theorems for Rus-Hicks-Rhoades-Jaggi hybrid combinational type mappings within orthogonal quasi-metric spaces. To illustrate and validate these results, an example is provided. Additionally, we highlight a practical application by connecting the theoretical findings to an orthogonal model in communication theory. Specifically, we relate the results to *space-time block coding (STBC)* in multiple-input multiple-output (MIMO) systems, where the fixed point solution represents the equilibrium state of iterative decoding, ensuring convergence to a stable codeword reconstruction even in the presence of channel disturbances. Moreover, we show that the Helmholtz equation with mixed boundary conditions possesses a unique fixed point. This framework has broad applicability: in acoustics, it models vibrations in air columns of closed-open tubes; in electromagnetic, it describes field distributions in wave-guides and resonant cavities; and in mechanics, it represents vibrations of beams with one fixed and one free end. Such formulations demonstrate how Helmholtz phenomena under mixed boundary value problems provide insights into wave propagation, resonance control, and system stability, thereby enriching both the theoretical understanding of fixed-point analysis and its engineering applications.

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Fixed point; RHR contraction mapping; orthogonal quasi-metric space; orthogonal system model; Helmholtz mixed BVPs

1 Introduction

The concept of a metric space, introduced by Frechet (1906), has been a foundation and backbone of the fixed point theory initiated by Banach (1922). Several metric space generalizations have since been made in different directions, such as b-metric space (Czerwik, 1993), partial metric space (Matthews, 1994) and quasi-metric space by Wilson (1931), which omitted the symmetric property and introduced fixed point results in quasi-metric spaces. Secelean et al. (2019) established new fixed point results in quasi-metric spaces and explored their applications in fractal theory. Fulga et al. (2020) established a fixed point theorem for hybrid contractions within the setting of quasi-metric spaces. Aydi et al. (2016) investigated fixed point results for α-implicit contractions in quasi-metric spaces and discussed their implications. Furthermore, Park (2023a, 2023b, 2024) contributed to the study by presenting fixed point results for Rus-Hicks-Rhoades contraction mappings in quasi-metric spaces.

Gordji et al. (2017) offered an expansion of the Banach contraction principle and produced findings about orthogonal sets. Additionally, they offered applications for their findings to ensure the exclusivity and existence of solutions for first-order differential equations. Bilgili Gungor (2022) extended an orthogonal p-contraction on orthogonal metric spaces. Javed et al. (2021) extended the results to an orthogonal partial b-metric spaces with an application. Uddin et al. (2021) proved an orthogonal results in m-metric spaces and an application to solve integral equations. Nazam et al. (2023) established existence theorems for (Ψ, Φ)-orthogonal interpolative contractions and demonstrated their application to fractional differential equations. Shahid et al. (2024) explored topological properties of q-rung orthopair fuzzy

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metric spaces, extending their theoretical framework. Nazam et al. (2023) analyzed iterative multivalued ⊥-preserving mappings and applied the results to fractional differential equations.

Several studies have explored related concepts, such as Rus-Hicks-Rhoades contraction mappings in quasi-metric spaces, as highlighted in the cited literature. While these works provide valuable foundational insights, none have specifically investigated the role of orthogonality in quasi-metric spaces. To address this gap, this study introduces a novel approach that integrates these elements, offering a comprehensive analysis of their theoretical properties and practical applications.

Younis and Abdou (2023), Younis et al. (2024, 2025), and Younis and Öztürk (2025) have advanced fixed point theory through several applications. They studied the Helmholtz equation with mixed boundary conditions using graphically controlled contractions to establish existence and uniqueness of solutions, addressing the gap in rigorous fixed point approaches for complex boundary-value problems. They also investigated best proximity points for multivalued mappings in metric spaces, providing a framework for approximate solutions where exact fixed points may not exist, with implications in applied mechanics and dynamic systems. Furthermore, they developed new proximal point theorems for diverse classes of nonlinear operators, expanding the applicability of classical methods to optimization and engineering problems. Finally, by introducing fuzzy contraction mappings, they provided a theoretical foundation for handling systems with uncertainty, enabling practical modeling and solution of engineering problems under imprecise conditions. Collectively, these contributions enhance both the analytical and applied dimensions of fixed point theory across mathematics, mechanics, and engineering sciences.

The work of Karapinar and Fulga (2019) introduced a hybrid contraction involving Jaggi-type mappings and established fixed point results within the framework of standard metric spaces, while their later paper (Karapinar & Fulga, 2022) extended this discussion to Jaggi-Meir-Keeler type contractions, focusing mainly on theoretical refinements without direct application to orthogonal structures. More recently, Shehu et al. (2025) studied combinational contractions and provided interesting applications, but their results were still confined to classical metric settings. None of these works considered the setting of orthogonal quasi-metric spaces or their connection to communication-theoretic models. Our manuscript fills this gap by formulating and proving new fixed point theorems for Rus-Hicks-Rhoades contractions in orthogonal quasi-metric spaces and illustrating their applicability to orthogonal system models motivated by spacetime block coding.

This work focuses on fixed point theorems for Rus-Hicks-Rhoades-Jaggi hybrid combinational type mapping in orthogonal quasi-metric spaces and extends conventional metric space results. Inspired by the contributions of distinguished researchers, including Gordji et al. (2017), Jleli and Samet (2012), Karapinar and Fulga (2019, 2022), Park (2023a, 2023b), and Shehu et al. (2025), we further illustrate the established theorems with a concrete example. Finally, we validate our findings through an orthogonal model of communication theory and application of Helmholtz equation with mixed boundary conditions to prove existence and uniqueness of solutions in orthogonal quasi-metric space.

2 Preliminaries

To ensure that the following results are apparent, we start with some essential terminology that are frequently employed in the study of fixed point theory.

Using Wilson (1931) and Gordji et al. (2017) concepts, we launch the study of orthogonal quasi-metric as follows:

Definition 2.1 *A function q: $X \times X \to [0, \infty)$, an orthogonal quasi-metric on a non-empty set X, satisfies the following axioms:*

1. $q(x, y) = 0$ if and only if $x = y$, and $x \perp y$;
2. $q(x, z) \leq q(x, y) + q(y, z)$, for all $x \perp y \in X$.

For $q(y, x)$, $q(z, y)$, and $q(z, x)$, similar relationships hold; this structure is referred to as an orthogonal quasi-metric space. If a quasi-metric on a set X satisfies the symmetry condition for all $x, y \in X$, then it is a metric. It is crucial to remember that a quasi-metric does not necessarily meet the requirements for being a metric. An orthogonal quasi-metric space is a pair $(X, \perp, q)$, also called a O-set, where q is the orthogonal quasi-metric.

The following are some examples of quasi-metric.

Example 2.1 *A metric derived from an orthogonal quasi-metric on $(X, \perp, q)$ forms a quasi-metric space, where the function $q: X \times X \to [0, \infty)$ is defined as follows:*

$$q(x, y) = \max\{q(x, y), q(y, x)\}.$$

A function is considered a metric on X if it distinguishes between distinct points x and y (i.e. $x \neq y$), but symmetry is not necessarily required.

Example 2.2 *Fulga et al. (2020) A quasi-metric space can be constructed from an orthogonal quasi-metric on $(X, \perp, q)$, where the function $q: X \times X \to [0, \infty)$ defines the quasi-metric.*

$$q(x, y) = \begin{cases} x - y, \ \forall\, x \geq y, \\ 2(x - y), \ \forall\, x < y, \end{cases}$$

is a quasi metric on X, for $x \neq y$, and not necessary symmetry.

Similarly, within the framework of orthogonal quasi-metric spaces, we outline the ideas of orthogonal completeness, continuity, and orthogonal sequences. The fundamental work first given by Jleli and Samet (2012) and later expanded upon by Gordji et al. (2017) is the basis for these ideas.

Definition 2.2 *The orthogonal quasi-metric space $(X, \perp, d)$ is assumed, and $x \in X$ and $\{x_k\}$ are sequences in X. $\{x_k\}$ converges to x if and only if*

$$\lim_{k\to\infty} q(x_k, x) = \lim_{k\to\infty} q(x, x_k) = 0, \quad x_k \perp x \vee x \perp x_k \ (\forall\, k \in \mathbb{N}).$$

Definition 2.3 *Think of a sequence $\{x_k\}$ in X that is orthogonal to the quasi-metric space $(X, \perp, q)$. If and only if there is a positive integer $N = N(\epsilon)$ such that the condition holds for every $\epsilon > 0$, we say that $\{x_k\}$ is a left O-Cauchy sequence.*

$$\lim_{k\to\infty} q(x_k, x_m) \leq \epsilon, \quad x_k \perp x_m \ \vee \ x_m \perp x_k \ (\forall\, k \geq m > N).$$

Definition 2.4 *Since $(X, \perp, q)$ is an orthogonal quasi-metric space, let $\{x_k\}$ represent a sequence in X. We assert that $\{x_k\}$ is a right O-Cauchy sequence if and only if there exists a positive integer $N = N(\epsilon)$ such that the subsequent condition holds for each $\epsilon > 0$.*

$$\lim_{k\to\infty} q(x_k, x_m) \leq \epsilon, \quad x_k \perp x_m \ \vee \ x_m \perp x_k \ (\forall\, k \geq m > N).$$

Definition 2.5 *A sequence $\{x_k\}$ in X and an orthogonal quasi-metric space $(X, \perp, d)$ are considered. For any $\epsilon > 0$, we assert that $\{x_k\}$ is a O-Cauchy sequence if and only if there exists a positive integer $N = N(\epsilon)$ such that the following condition holds.*

$$\lim_{k\to\infty} q(x_k, x_m) \leq \epsilon, \quad x_k \perp x_m \ \vee \ x_m \perp x_k \ (\forall\, k \geq m > N).$$

Definition 2.6 *(Gordji et al., 2017) The binary relation $(X, \perp)$ is assumed to be based on the Cartesian product $X \times X$. Accordingly,*

$$\forall\, x \in X, \quad x_0 \perp x \vee \forall\, y \in X,\, y \perp x_0.$$

is true if an element $x_0 \in X$ and $(X, \perp)$ form an orthogonal set (O-set). x_0 is referred to as an orthogonal element. It is possible for an orthogonal set to contain multiple orthogonal elements.

For completeness of O-quasi-metric, we have the following:

Definition 2.7 *Let $(X, \perp, q)$ be an orthogonal quasi-metric space, we say that*

(i) In order for the structure $(X, \perp, q)$ to be regarded as left complete, each left O-Cauchy sequence in X must converge.
(ii) If and only if every right O-Cauchy sequence in X converges, then $(X, \perp, q)$ is right complete.
(iii) The completeness of $(X, \perp, q)$ depends on the convergence of each O-Cauchy sequence in X.

Definition 2.8 *Examine the orthogonal quasi-metric space $(X, \perp, d)$. If a O-sequence $\{x_k\}$ approaches the limit x, then $Tx_k \to Tx$ as $k \to \infty$. This implies that a function $T: X \to X$ is thought to have orthogonality continuity (O-continuity) at x.*

Definition 2.9 *If $x \perp y$ implies $Tx \perp Ty$ for every $x, y \in X$, then the state of a self mapping T on O-quasi metric space is orthogonality ($\perp$-preserving).*

Example 2.3 *(Gordji et al., 2017) Under the assumption that $X = [0,1]$, the Euclidean metric is defined on the set X. We define the connection $x \perp y$ for any elements $x, y \in X$ if both x and y are members of the set $\{x, y\}$. An additional assumption is that the mapping $T: X \to X$ is defined by*

$$Tx = \begin{cases} \frac{x}{2}, \, x \in \mathbb{Q} \cap X, \\ 0, \, x \in \mathbb{Q}^c \cap X. \end{cases} \tag{2.1}$$

Then, T is a $\perp$-preserving mappings.

The subsequent findings constitute preliminary results.

Gordji et al. (2017) introduced the following definition and theorem in the context of orthogonal metric spaces.

Definition 2.10 *Consider the orthogonal metric spaces $0 < \lambda < 1$ and $(X, \perp, d)$. A mapping $T: X \to X$ is an orthogonal contraction with Lipschitz constant λ if the following condition is true for all $x, y \in X$ such that $x \perp y$.*

$$d(Tx, Ty) \leq \lambda d(x, y).$$

Theorem 2.1 *Suppose that $(X, \perp, d)$ and $0 < \lambda < 1$ are in complete metric spaces with orthogonal structures. Let $T: X \to X$ be an orthogonal contraction with an orthogonality $\perp$ and a Lipschitz constant λ. $x^\star \in X$ is the only fixed point for T. Moreover, T is a Picard operator, meaning that for every $x \in X$, the sequence $\{T^k x\}$ converges to $x^\star$ as $k \to \infty$.*

Theorem 2.2 Rus (1973) *Let $T: X \to X$ be a continuous self-map of a complete metric space (X, d) that satisfies:*

$$d(Tx, T^2x) \leq \alpha d(x, Tx), \forall\, x \in X,$$

where $0 < \alpha < 1$. Then T has a fixed point.

The Rus-Hicks-Rhoades (RHR) contraction was illustrated by Park (2023a, 2023b, 2024); Suzuki (2001) in the following way:

Theorem 2.3 *Let $T: X \to X$ be a map that satisfies the RHR property, and let (X, q) be a quasi-metric space.*

$$q(Tx, T^2x) \leq \beta q(x, y), \forall x \in X, \tag{2.2}$$

where $0 < \beta < 1$.

(i) There is a point $x_0 \in X$ for each $x \in X$ indicates that X is T-orbitally complete.

$$\lim_{i \to \infty} T^k x = x_0, \tag{2.3}$$

and

$$q(T^k x, x_0) \leq \frac{\beta^k}{1-\beta} d(x, Tx), \forall\, i = 1, 2, \ldots$$

$$q(T^k x, x_0) \leq \frac{\beta^k}{1-\beta} d(T^{k-1}x, T^k x), \forall\, i = 1, 2, \ldots$$

(ii) x_0 is a fixed point of T and similarly
(iii) At $x_0 \in X$, $T: X \to X$ is orbit-ally continuous.

The definition that follows was introduced by Samet et al. (2012).

Definition 2.11 *The set Ψ can be thought of as The collection of functions that do not decrease, $\psi: [0, \infty) \to [0, \infty)$, that*

$$\sum_{k=1}^{\infty} \psi^k(s) < \infty$$

for all $s > 0$. It is easy to see that $\psi(s) < s$ for all $s \in [0, \infty)$.

Karapinar and Fulga (2019, 2022) described a hybrid type in the context of complete metric spaces as follows:

Definition 2.12 *The Jaggi hybrid contraction is a self-mapping T on a metric space (X, d) if $\psi \in \Psi$ exists such that*

$$d(Tx, Ty) \leq \psi(M_T^s(x, y)),$$

for all $x, y \in X$, where $s \geq 0$ and $\mu_i \geq 0$, $i = 1, 2$, such that $\mu_1 + \mu_2 = 1$ with $F_T(X) = \{z \in X: Tz = z\}$ and

$$M_T^s(x, y) = \begin{cases} \left[\mu_1\left(\frac{d(x, Ty), d(y, Ty)}{d(x, y)}\right)^s + \mu_2(d(x, y))^s\right]^{\frac{1}{s}}, x \neq K, \\ [d(x, Tx)]^{\mu_1}[d(y, Ty)]^{\mu_2}, x, y \in X \setminus F_T(X). \end{cases} \tag{2.4}$$

Theorem 2.4 *There is a fixed point in X for T on a metric space (X, d) if T is a Jaggi-type hybrid contraction. Furthermore, the sequence $\{T^k x_0\}$ converges to the fixed point x for any starting point $x_0 \in X$.*

The hybrid combinational contraction was introduced by Shehu et al. (2025) as follows:

Definition 2.13 *Assume that* (X, d) *is a metrics space. If* $\delta > 0$ *is such that a mapping* $T: X \to X$ *is combinational for every* $\epsilon > 0$ *and for every pair of unique points* $x, y \in X$.

$$\epsilon < d(x, y) < \epsilon + \delta \Rightarrow \rho(x, y) d(Tx, Ty) \leq \epsilon, \tag{2.5}$$

where

$$\rho(x, y) d(Tx, Ty) < \max\{d(x, y), \frac{1}{2}[d(x, Ty) + d(y, Ty)],$$

$$\frac{1}{2}[d(x, Ty) + d(y, Tx)], M_T^s(x, y)\}.$$

Theorem 2.5 *A continuous combinational contraction on a complete metric space* (X, d) *is denoted by* $T: X \to X$. *This indicates that* X *contains a fixed point for* T.

3 Main results

The following definition opens this section.

Definition 3.1 *Let* (X, q) *be an orthogonal quasi-metric space. A mapping* $T: X \to X$ *is called a hybrid combinational Rus-Hicks-Rhoades-Ćirić type mapping if, for all distinct points* $x, y \in X$, *there exist constants* $\gamma < 1$, $p \geq 0$, *and* $k_i \geq 0$ *for* $i = 1,2,3,4$, *with* $\sum_{i=1}^{4} k_i < 1$, *and the following conditions hold:*

(i) The mapping satisfies the inequality

$$q(Tx, T^2x) \leq \max\{\min\{q(x, Ty), q(y, Tx)\}, M_T^p(x, y)\}. \tag{3.1}$$

where

$$M_T^p(x, y) = \begin{cases} [k_1(q(x, y))^p + k_2(q(x, Tx))^p + k_3(q(y, Ty))^p \\ \quad + k_4(\frac{q(x, Ty) + q(y, Tx)}{2})^p]^{\frac{1}{p}}, \text{ for } p \geq 0,\ x, y \in X; \\ \gamma [q(x, y)]^{k_1} [q(x, Tx)]^{k_2}.\ [q(y, Ty)]^{k_3}, \\ \quad \text{for } p = 0,\ k_4 = 0 x, y \in X. \end{cases} \tag{3.2}$$

(ii) There exist constants $\Delta = \left[\frac{2^p(k_1 + k_2) + k_4}{2^p(1 - k_3) - k_4}\right]^{\frac{1}{p}} < 1$ and $\delta = \left[\frac{2^p(k_1 + k_3) + k_4}{2^p(1 - k_2) - k_4}\right]^{\frac{1}{p}} < 1$.

Theorem 3.1 *Let* (X, q) *be a complete orthogonal quasi-metric space and let* $T: X \to X$ *be a hybrid combinational contraction in the sense of Rus-Hicks-Rhoades-Ćirić. Then* T *admits a fixed point in* X.

Proof. Let $X_k \in TX_{k-1}$, and let x_0 be an arbitrary point in X. The following is what orthogonality means:

$$T^k x_0 \perp T^{k+1} x_0 \quad \text{or}$$

The sequence $\{T^k x_0\}$ is orthogonal, as implied by

$$T^{k+1}x_0 \perp T^k x_0$$

for all $k \in \mathbb{N} \cup \{0\}$. We also have the condition

$$x_k \perp Tx_k \quad \text{or} \quad Tx_k \perp x_k$$

for all $k \in \mathbb{N} \cup \{0\}$.

For every $k \in \mathbb{N}$, define the sequence $\{x_k\}$ in X using the recurrence $x_{k+1} = Tx_k$. Then, for every $k \in \mathbb{N} \cup \{0\}$, we obtain $x_k = T^k x_0$.

The proof is complete if $x_k = T^k x_0 = T^{k+1}x_0 = x_{k+1}$ for some $k \in \mathbb{N} \cup \{0\}$ is true, and $T^k x_0 = x_k$ is a fixed point of T.

Conversely, we declare that $q(Tx_k, Tx_{k+1}) > 0$ if $x_k \neq x_{k+1}$ for every $k \in \mathbb{N} \cup \{0\}$. Given that T maintains orthogonality, the sequence $\{T^k x_0\}$ is orthogonal since

$$T^k x_0 \perp T^{k+1}x_0 \text{ or } T^{k+1}x_0 \perp T^k x_0$$

for all $k \in \mathbb{N} \cup \{0\}$ is orthogonal.

From this, we can conclude that $x_k = Tx_0$ and $x_{k+1} = T^2 x_0$. Given $x = x_{k-1}$ and $y = x_k$, we may use the contractive inequality (3.1) to get

$$q(x_k, x_{k+1}) \leq \max\{\min\{q(x_{k-1}, Tx_k), q(x_k, Tx_{k-1})\}, M_T^p(x_{k-1}, x_k)\}. \tag{3.3}$$

Two cases need our investigation:

Case 3.1.1 *From* (3.2) *for* $p \geq 0$, *we have*

$$\begin{aligned} &q(x_k, x_{k+1}) \\ &\leq \max\Bigg\{\min\{q(x_{k-1}, Tx_k), q(x_k, Tx_{k-1})\}, \\ &\quad \left[k_1(q(x_{k-1}, x_k))^p + k_2(q(x_{k-1}, Tx_{k-1}))^p + k_3((q(x_k, Tx_k))^p + k_4\left(\frac{q(x_{k-1}, Tx_k) + q(x_k, Tx_{k-1})}{2}\right)^p\right]^{\frac{1}{p}}\Bigg\}, \\ &\leq \max\Bigg\{\min\{q(x_{k-1}, x_{k+1}), q(x_k, x_k)\}, \\ &\quad \left[k_1(q(x_{k-1}, x_k))^p + k_2(q(x_{k-1}, x_k))^p + k_3(q(x_k, x_{k+1})^p + k_4\left(\frac{q(x_{k-1}, x_{k+1}) + q(x_k, x_k)}{2}\right)^p\right]^{\frac{1}{p}}\Bigg\}. \end{aligned} \tag{3.4}$$

Applying $(W2)$ in (3.4), we obtain

$$\begin{aligned} q(x_k, x_{k+1}) \leq \max\{&\min\{[q(x_{k-1}, x_k) + q(x_k, x_{k+1})], q(x_k, x_k)\}, \\ &[k_1(q(x_{k-1}, x_k))^p + k_2(q(x_{k-1}, x_k))^p + k_3(q(x_k, x_{k+1}))^p \\ &+ k_4\left(\frac{q(x_{k-1}, x_k) + q(x_k, x_{k+1})}{2}\right)^p\Bigg]^{\frac{1}{p}}\Bigg\}, \end{aligned}$$

$$q(x_k, x_{k+1}) \leq \max\left\{0, \left[\left(k_1 + k_2 + \frac{k_4}{2^p}\right)(q(x_{k-1}, x_k))^p + \right.\right.$$

$$\left.\left.\left(k_3 + \frac{k_4}{2}\right)(q(x_k, x_{k+1}))^p\right]^{\frac{1}{p}}\right\},$$

$$q(x_k, x_{k+1}) \leq \left[\left(k_1 + k_2 + \frac{k_4}{2^p}\right)(q(x_{k-1}, x_k))^p + \left(k_3 + \frac{k_4}{2^p}\right)(q(x_k, x_{k+1}))^p\right]^{\frac{1}{p}},$$

$$(q(x_k, x_{k+1}))^p \leq \left(k_1 + k_2 + \frac{k_4}{2^p}\right)(q(x_{k-1}, x_k))^p + \left(k_3 + \frac{k_4}{2^p}\right)(q(x_k, x_{k+1}))^p.$$

Consequently, we get

$$\begin{aligned} \left(1 - k_3 - \frac{k_4}{2^p}\right)(q(x_k, x_{k+1}))^p &\leq \left(k_1 + k_2 + \frac{k_4}{2^p}\right)(q(x_{k-1}, x_k))^p, \quad (q(x_k, x_{k+1}))^p \\ &\leq \frac{2^p(k_1 + k_2) + k_4}{2^p(1 - k_3) - k_4}(q(x_{k-1}, x_k))^p, \quad q(x_k, x_{k+1}) \\ &\leq \left[\frac{2^p(k_1 + k_2) + k_4}{2^p(1 - k_3) - k_4}\right]^{\frac{1}{p}} q(x_{k-1}, x_k), \quad q(x_k, x_{k+1}) \leq \Delta q(x_{k-1}, x_k), \end{aligned} \tag{3.5}$$

for all $k \in \mathbb{N}$, where $\Delta = \left[\frac{2^p(k_1+k_2)+k_4}{2^p(1-k_3)-k_4}\right]^{\frac{1}{p}} < 1$.

Repeating k-times by mathematical induction, using (3.5), we get

$$q(x_k, x_{k+1}) \leq \Delta^k q(x_{k-1}, x_k). \tag{3.6}$$

Then, by (3.10), $\{x_k\}$ is a right-Cauchy sequence in (X, q).

In a similar manner, by setting $x = x_k$, $y = x_{k-1}$, and $T^2 x_k = T x_{k-1} = x_k$, we apply the contractive inequality (3.1) to derive:

$$q(x_{k+1}, x_k) \leq \max\{\min\{q(x_k, Tx_{k-1}), q(x_{k-1}, Tx_k)\}, M_T^p(x_k, x_{k-1})\}. \tag{3.7}$$

$$\begin{aligned} q(x_{k+1}, x_k) \leq\ & \max\{\min\{q(x_k, Tx_{k-1}), q(x_{k-1}, Tx_k)\}, M_T^p(x_k, x_{k-1})\}.\ q(x_{k+1}, x_k) \\ \leq\ & \max\Bigg\{\min\{q(x_k, Tx_{k-1}), q(x_{k-1}, Tx_k)\}, \\ & \left[k_1(q(x_k, x_{k-1}))^p + k_2(q(x_k, Tx_k))^p + k_3((q(x_{k-1}, Tx_{k-1}))^p + k_4\left(\frac{q(x_k, Tx_{k-1}) + q(x_{k-1}, Tx_k)}{2}\right)^p\right]^{\frac{1}{p}}\Bigg\}, \\ \leq\ & \max\Bigg\{\min\{q(x_k, x_k), q(x_{k-1}, x_{k+1})\}, \\ & \left[k_1(q(x_k, x_{k-1}))^p + k_2(q(x_k, x_{k+1}))^p + k_3(q(x_{k-1}, x_k))^p + k_4\left(\frac{q(x_k, x_k) + q(x_{k-1}, x_{k+1})}{2}\right)^p\right]^{\frac{1}{p}}\Bigg\}. \end{aligned} \tag{3.8}$$

By substituting condition $(W2)$ into inequality (3.8), we get

$$q(x_{k+1}, x_k) \leq \max\{\min\{q(x_k, x_k), [q(x_{k-1}, x_k) + q(x_k, x_{k+1})]\},$$

$$[k_1(q(x_k, x_{k-1}))^p + k_2(q(x_k, x_{k+1}))^p + k_3(q(x_{k-1}, x_k))^p$$

$$+ k_4(\frac{q(x_{k-1}, x_k) + q(x_k, x_{k+1})}{2})^p]^{\frac{1}{p}}\}.$$

Assume that $q(x_{k-1}, x_k) < q(x_k, x_{k-1})$ and $q(x_k, x_{k+1}) < q(x_{k+1}, x_k)$ in the above inequality, we get

$$q(x_{k+1}, x_k) \leq \max\{0, [\left(k_1 + k_3 + \frac{k_4}{2^p}\right)(q(x_k, x_{k-1}))^p +$$

$$\left(k_2 + \frac{k_4}{2^p}\right)(q(x_{k+1}, x_k))^p\Big]^{\frac{1}{p}}\Big\},$$

$$q(x_{k+1}, x_k) \leq \left[\left(k_1 + k_3 + \frac{k_4}{2^p}\right)(q(x_k, x_{k-1}))^p + \left(k_2 + \frac{k_4}{2^p}\right)(q(x_{k+1}, x_k))^p\right]^{\frac{1}{p}},$$

$$(q(x_{k+1}, x_k))^p \leq \left(k_1 + k_3 + \frac{k_4}{2^p}\right)(q(x_k, x_{k-1}))^p + \left(k_2 + \frac{k_4}{2^p}\right)(q(x_{k+1}, x_k))^p.$$

As a result, we obtain

$$\begin{aligned}\left(1 - k_2 - \frac{k_4}{2^p}\right)(q(x_{k+1}, x_k))^p &\leq \left(k_1 + k_3 + \frac{k_4}{2^p}\right)(q(x_k, x_{k-1}))^p, \quad (q(x_{k+1}, x_k))^p \\ &\leq \frac{2^p(k_1 + k_3) + k_4}{2^p(1 - k_2) - k_4}(q(x_k, x_{k-1}))^p, \quad q(x_{k+1}, x_k) \\ &\leq \delta q(x_k, x_{k-1}), \quad q(x_{k+1}, x_k) \leq \delta^{\frac{1}{p}} q(x_k, x_{k-1}),\end{aligned} \tag{3.9}$$

for all $k \in \mathbb{N}$, where $\delta = \left[\frac{2^p(k_1 + k_3) + k_4}{2^p(1 - k_2) - k_4}\right]^{\frac{1}{p}} < 1$.

Repeating the above procedure k-times, using (3.5), we get

$$q(x_{k+1}, x_k) \leq \delta^k q(x_1, x_0). \tag{3.10}$$

Then, by (3.10), $\{x_k\}$ is a left-Cauchy sequence in (X, q).

Consequently, the sequence x_k is Cauchy throughout the entire quasi-metric space (X, q) since it is Cauchy from both the left and the right.

We then demonstrate the Cauchy nature of the sequence $\{x_k\}$ in a complete quasi-metric space. If l > k, then $k, l \geq \mathbb{N}$. The inequality (3.10) and $(W2)$ yield

$$\begin{aligned} q(x_k, x_l) &\leq q(x_k, x_{k+1}) + q(x_{k+1}, x_{k+2}) + q(x_{k+2}, x_{k+3}) + \ldots + q(x_{l-1}, x_l), \\ q(x_k, x_l) &\leq [\Delta^k + \Delta^{k+1} + \Delta^{k+2} + \Delta^{k+3} + \ldots + \Delta^{l-1}]q(x_0, x_1), \\ q(x_k, x_l) &\leq \Delta^k[1 + \Delta^1 + \Delta^2 + \Delta^3 + \ldots + \Delta^{l-1-k}]q(x_0, x_1), \\ q(x_k, x_l) &\leq \tfrac{\Delta^k}{1-\Delta} q(x_0, x_1). \end{aligned}$$

As $k \to \infty$, $q(x_k, x_l) = 0$. This proves that x_k is a Cauchy sequence.

To the extent that X is a full quasi-metric space, the sequence $\{x_k\}$ converges to $\aleph \in X$. Then, we show that $\aleph$ is a fixed point of T. $(x_k, \aleph) \in X$, of course. With $x = x_k$ and $K = \aleph$, we apply the contractive condition (3.1) to get

$$q(x_k, \aleph) \leq \max\left\{\min\{q(x_k, T\aleph), q(\aleph, Tx_k)\},\right.$$
$$\left.\left[k_1(q(x_k, \aleph))^p + k_2(q(x_k, Tx_k))^p + k_3(q(\aleph, T\aleph)^p, \quad +k_4(\frac{q(x_k, T\aleph) + q(\aleph, Tx_k)}{2})^p\right]^{\frac{1}{p}}\right\}, \tag{3.11}$$

for all $k \in \mathbb{N}$.

On taking limit as $k \to \infty$ on both sides of (3.11), yields to

$$q(\aleph, \aleph) \leq \max\{\min\{q(\aleph, T\aleph), q(\aleph, T\aleph)\},$$

$$[k_1(q(\aleph, \aleph))^p + k_2(q(\aleph, T\aleph)^p + k_3(q(\aleph, T\aleph)^p$$

$$+k_4(\frac{q(\aleph, T\aleph) + q(\aleph, T\aleph)}{2})^p]^{\frac{1}{p}}\},$$

$$q(\aleph, \aleph) \leq \max\{q(\aleph, T\aleph), [(k_2 + k_3 + k_4)(q(\aleph, T\aleph))^p]^{\frac{1}{p}}\},$$

$$q(\aleph, \aleph) \leq q(\aleph, T\aleph),$$

$$0 \leq q(\aleph, T\aleph),$$

that is $\aleph = T\aleph$, thus $\aleph$ is a fixed point of T.

If $\aleph^*$ and $\aleph$ are two distinct fixed points with an orthogonal relationship $\aleph \perp \aleph^*$, applying equation (3.1) with $x = \aleph$ and $K = \aleph^*$, we obtain:

$$q(\aleph, \aleph^*) \leq \max\{\min\{q(\aleph, T\aleph^*), q(\aleph^*, T\aleph)\},$$

$$[k_1(q(\aleph, \aleph^*))^p + k_2(q(\aleph, T\aleph))^p + k_3(q(\aleph^*, T\aleph^*)^p$$

$$+k_4(\frac{q(\aleph, T\aleph^*) + q(\aleph^*, T\aleph)}{2})^p]^{\frac{1}{p}}\},$$

$$q(\aleph, \aleph^*) \leq \max\{q(\aleph^*, \aleph), [(k_1 + k_4)(q(\aleph, \aleph^*))^p]^{\frac{1}{p}}\},$$

$$q(\aleph, \aleph^*) \leq \max\{q(\aleph^*, \aleph), (k_1 + k_4)^{\frac{1}{p}}(q(\aleph, \aleph^*))\},$$

$$q(\aleph, \aleph^*) \leq q(\aleph^*, \aleph).$$

The above inequality satisfies when $\aleph = \aleph^*$. Hence $\aleph$ is a unique fixed point of T.

Case 3.1.2 *For $p = 0$ and $\gamma < 1$, by substituting $x = x_{k-1}$ and $y = x_k$, by using the contractive inequality (3.1), we get*

$$\begin{aligned} q(Tx_{k-1}, T^2x_{k-1}) \leq\ & \max\{\min\{q(x_{k-1}, Tx_k), q(x_k, Tx_{k-1})\},\ \gamma[q(x_{k-1}, x_k)]^{k_1}[q(x_{k-1}, Tx_{k-1})]^{k_2}. \\ & [q(x_k, Tx_k)]^{K_3}\}, q(x_k, x_{k+1}) \leq \max\{\min\{q(x_{k-1}, x_{k+1}), q(x_k, x_k)\}, \\ & \gamma[q(x_{k-1}, x_k)]^{k_1}[q(x_{k-1}, x_k)]^{k_2}.\ [q(x_k, x_{k+1})]^{K_3}\}, q(x_k, x_{k+1}) \\ \leq\ & \max\{0, \gamma[q(x_{k-1}, x_k)]^{k_1+k_2}.\ [q(x_k, x_{k+1})]^{K_3}\}, q(x_k, x_{k+1}) \leq \gamma[q(x_{k-1}, x_k)]^{k_1+k_2}. \\ & [q(x_k, x_{k+1})]^{K_3}. \end{aligned} \tag{3.12}$$

Let $q(x_k, x_{k+1}) < q(x_{k-1}, x_k)$ in (3.12), we obtain

$$q(x_k, x_{k+1}) \leq \gamma [q(x_{k-1}, x_k)]^{k_1+k_2+k_3},$$

$$q(x_k, x_{k+1}) \leq \gamma q(x_{k-1}, x_k) \leq \gamma^2 q(x_{k-1}, x_k) \ldots \leq \gamma^k q(x_0, x_1),$$

which shows that $\{x_k\}$ is a right-Cauchy sequence in (X, q).

By replacing $x = x_k$, $y = x_{k-1}$, and using the relation $T^2 x_k = T x_{k-1}$, along with the application of the contractive inequality (3.1), we follow a similar method as previously to form a left-Cauchy sequence.

$$q(x_{k+1}, x_k) \leq \gamma q(x_k, x_{k-1}) \leq \gamma^2 q(x_k, x_{k-1}) \ldots \leq \gamma^k q(x_1, x_0).$$

The other steps of this proof follows a similar proof of case 1. Hence the proof is completed. □

Next, we investigates the following results.

Theorem 3.2 *Let $(X, \perp, d)$ be an orthogonal quasi-metric space, and let $T: X \to X$ be a mapping of the orthogonal RHR-Jaggi hybrid combinational type such that*

$$q(Tx, T^2x) \leq \mu q(x, Tx) + \vartheta q(x, T^2x) + M_T^z(x, y), \forall\, x \in X, \tag{3.13}$$

where

$$M_T^z(x, y) = \begin{cases} \left[k_1 (q(x, y))^z + k_2 \left(\frac{q(x, Ty) + q(y, Tx)}{2}\right)^z\right]^{\frac{1}{z}}, \, for z \geq 0,\, x, y \in X; \\ \left[\frac{q(x, Tx) + q(y, Tx)}{q(x, y) + q(y, Ty)}\right]^{k_1} [q(x, y)]^{k_2}, \, for z = 0,\, x, y \in X, \end{cases} \tag{3.14}$$

and $0 < \mu + \vartheta < 1$.

(i) If X is T complete then, for each $x \in X$, there exists a point $x_0 \in X$ such that

$$\lim_{k \to \infty} T^k x = x_0, \tag{3.15}$$

and

$$q(x_k, x_{k+1}) \leq \frac{\zeta^k}{1 - \zeta} q(x_0, x_1), \forall\, k = 1, 2, \ldots$$

where $\zeta = \frac{(\mu + \vartheta + (k_1 + k_2)^{\frac{1}{z}})}{1 - \vartheta}$.

(ii) x_0 is a fixed point of T.

Then T have at least one fixed point.

Proof. Let the sequence $\{x_k\}$ in orthogonal set $(X, \perp)$ be defined by

$$x_0 \in X: \forall\, x \in X,\, x \perp x_0 \;\vee\; x \in X,\, x_0 \perp x.$$

Accordingly

$$x_0 \perp T x_0 \;\vee\; T x_0 \perp x_0.$$

We define a Picard sequence in the following manner.

$$x_1 = T x_0$$

$$x_2 = Tx_1 = T^2x_0$$
$$\ldots\ldots$$
$$x_k = Tx_{k-1} = T^kx_0 \ \ \forall \ k \in \mathbb{N}.$$

Delegate $x_k = Tx_{k-1}$ to $x_0 \in X$. If there exists some $k \in \mathbb{N} \cup \{0\}$ such that $x_k = x_{k+1}$, then $Tx_k = x_k$ must hold. Since x_k is a fixed point of T, we conclude the result, thereby completing the proof.

In all other cases, if $x_k \leq x_{k+1}$, then for any $k \in \mathbb{N} \cup \{0\}$, it follows that $q(x_k, x_{k+1}) > 0$. Since T is $\perp$-preserving, we obtain

$$x_k \perp x_{k+1} \vee x_{k+1} \perp x_k.$$

This confirms that $\{x_k\}$ forms an orthogonal sequence. Setting $x = x_{k-1}$ and $y = x_k$, and applying inequality (3.13), we derive the following result.

Case 3.2.1 *For $z \geq 0$, it holds that*

$$\begin{aligned} q(Tx_{k-1}, T^2x_{k-1}) &\leq \mu q(x_{k-1}, Tx_{k-1}) + \vartheta q(x_{k-1}, T^2x_{k-1}) \\ &\quad + \left[k_1(q(x_{k-1}, x_k))^z + k_2\left(\frac{q(x_{k-1}, Tx_k) + q(x_k, Tx_{k-1})}{2}\right)^z\right]^{\frac{1}{z}}, \\ q(x_k, x_{k+1}) &\leq \mu q(x_{k-1}, x_k) + \vartheta q(x_{k-1}, x_{k+1}) \\ &\quad + \left[k_1(q(x_{k-1}, x_k))^z + k_2\left(\frac{q(x_{k-1}, x_{k+1}) + q(x_k, x_k)}{2}\right)^z\right]^{\frac{1}{z}}, \\ q(x_k, x_{k+1}) &\leq (\mu + \vartheta)q(x_{k-1}, x_k) + \vartheta q(x_k, x_{k+1}) \\ &\quad + \left[k_1(q(x_{k-1}, x_k))^z + k_2\left(\frac{q(x_{k-1}, x_k) + q(x_k, x_{k+1})}{2}\right)^z\right]^{\frac{1}{z}}. \end{aligned} \tag{3.16}$$

Suppose that $\left(\frac{q(x_{k-1}, x_k) + q(x_k, x_{k+1})}{2}\right) \leq q(x_{k-1}, x_k)$, using (3.16) leads to

$$q(x_k, x_{k+1}) \leq (\mu + \vartheta)q(x_{k-1}, x_k) + \vartheta q(x_k, x_{k+1}) + [k_1(q(x_{k-1}, x_k))^z$$
$$+k_2(q(x_{k-1}, x_k)^z]^{\frac{1}{z}},$$
$$q(x_k, x_{k+1}) \leq (\mu + \vartheta)q(x_{k-1}, x_k) + \vartheta q(x_k, x_{k+1}) +$$
$$[(k_1 + k_2)(q(x_{k-1}, x_k))^z]^{\frac{1}{z}},$$
$$q(x_k, x_{k+1}) \leq (\mu + \vartheta)q(x_{k-1}, x_k) + \vartheta q(x_k, x_{k+1}) +$$
$$(k_1 + k_2)^{\frac{1}{z}}(q(x_{k-1}, x_k)),$$
$$(1 - \vartheta)q(x_k, x_{k+1}) \leq (\mu + \vartheta + (k_1 + k_2)^{\frac{1}{z}})q(x_{k-1}, x_k),$$
$$q(x_k, x_{k+1}) \leq \frac{(\mu + \vartheta + (k_1 + k_2)^{\frac{1}{z}})}{1 - \vartheta}q(x_{k-1}, x_k),$$
$$q(x_k, x_{k+1}) \leq \zeta q(x_{k-1}, x_k),$$

where $\zeta = \frac{(\mu + \vartheta + (k_1 + k_2)^{\frac{1}{z}})}{1 - \vartheta}$, we deduce the following

$$q(x_k, x_{k+1}) \leq \zeta q(x_{k-1}, x_k) \leq \zeta^2 q(x_{k-1}, x_k) \leq \ldots \leq \zeta^k q(x_0, x_1).$$

Implies that $\{x_k\}$ is a right-Cauchy sequence.We can establish that

$$q(x_k, x_{k+1}) \leq \frac{\zeta^k}{1-\zeta} q(x_0, x_1), \ \forall \ k = 1, 2, \ldots$$

where $\zeta = \frac{(\mu + \vartheta + (k_1+k_2)^{\frac{1}{z}})}{1-\vartheta}$, is a complete right-Cauchy sequence.

Case 3.2.2 *For all $z \geq 0$, let $x = x_k$ and $y = x_{k-1}$ with $T^2x_k = Tx_{k-1}$. By applying inequality (3.13), we obtain the following result.*

$$\begin{aligned} q(Tx_k, T^2x_k) &\leq \mu q(x_k, Tx_k) + \vartheta q(x_k, x_k) \\ &\quad + \left[k_1(q(x_k, x_{k-1}))^z + k_2\left(\frac{q(x_k, Tx_{k-1}) + q(x_{k-1}, Tx_k)}{2}\right)^z\right]^{\frac{1}{z}}, \\ q(x_{k+1}, x_k) &\leq \mu q(x_k, x_{k+1}) + \vartheta q(x_k, x_k) \\ &\quad + \left[k_1(q(x_k, x_{k-1}))^z + k_2\left(\frac{q(x_k, x_k) + q(x_{k-1}, x_{k+1})}{2}\right)^z\right]^{\frac{1}{z}}, \\ q(x_{k+1}, x_k) &\leq (\mu + \vartheta) q(x_k, x_{k+1}) + \left[k_1(q(x_k, x_{k-1}))^z + k_2\left(\frac{q(x_{k-1}, x_k) + q(x_k, x_{k+1})}{2}\right)^z\right]^{\frac{1}{z}}. \end{aligned} \tag{3.17}$$

Suppose that $\left(\frac{q(x_{k-1}, x_k) + q(x_k, x_{k+1})}{2}\right) \leq q(x_k, x_{k-1})$ and $q(x_{k-1}, x_k) < q(x_k, x_{k-1})$, from (3.17). we get

$$q(x_{k+1}, x_k) \leq (\mu + \vartheta) q(x_{k+1}, x_k) + [k_1(q(x_k, x_{k-1}))^z + k_2(q(x_k, x_{k-1})^z]^{\frac{1}{z}},$$

$$q(x_{k+1}, x_k) \leq (\mu + \vartheta) q(x_{k+1}, x_k) + [(k_1 + k_2)(q(x_k, x_{k-1}))^z]^{\frac{1}{z}},$$

$$q(x_{k+1}, x_k) \leq (\mu + \vartheta) q(x_{k+1}, x_k) + (k_1 + k_2)^{\frac{1}{z}}(q(x_k, x_{k-1})),$$

$$(1 - \mu - \vartheta) q(x_{k+1}, x_k) \leq (k_1 + k_2)^{\frac{1}{z}} q(x_{k-1}, x_k),$$

$$q(x_{k+1}, x_k) \leq \frac{(k_1 + k_2)^{\frac{1}{z}}}{1 - \mu - \vartheta} q(x_k, x_{k-1}),$$

$$q(x_{k+1}, x_k) \leq \eta q(x_k, x_{k-1}),$$

where $\eta = \frac{(k_1+k_2)^{\frac{1}{z}}}{1-\mu-\vartheta}$, thus, we obtain

$$q(x_{k+1}, x_k) \leq \zeta q(x_k, x_{k-1}) \leq \eta^2 q(x_k, x_{k-1}) \leq \ldots \leq \eta^k q(x_1, x_0).$$

Implies that $\{x_k\}$ is a left-Cauchy sequence.

Eventually, we can show that

$$q(x_{k+1}, x_k) \leq \frac{\eta^k}{1-\eta} q(x_1, x_0), \ \forall \ k = 1, 2, \ldots$$

where $\eta = \frac{(k_1+k_2)^{\frac{1}{z}}}{1-\mu-\vartheta}$, is a complete left-Cauchy sequence. □

The subsequent steps for establishing the Cauchy sequence and proving uniqueness proceed in the same manner as outlined in the proof of Theorem 3.1.

Finally, we present the following corollary, which extends the results established in Theorem 3.1.

Corollary 3.1 *Let $(X, \perp, q)$ be an orthogonal quasi-metric space and ψ: $(0, \infty) \to (0, \infty)$ is a function. A ψ-orthogonal, Reich-Rus-Ciri'c hybrid combinational type contraction is defined as a mapping T: $X \to X$ such that*

$$q(Tx, T^2x) \leq \psi(\tau \mathscr{M}_q(x, y)), \tag{3.18}$$

where

$$\mathscr{M}_q(x, y) = \max\{\min\{q(x, y), q(x, Ty), q(y, Tx)\}, M_T^s(x, y)\} \tag{3.19}$$

$\forall\ x \in X$ and $(x, y) \in \perp$. If there exists $x \in X$ and $T(X)$ is $\perp$-connected or $\perp$-preserved, then T has a unique fixed point.

Proof. The proof of this corollary follows similar steps as those in Theorem 3.1 and utilizes inequality (3.18). Thus, the proof is complete. □

To highlight the novelty of our results, we present the following example corresponding to Theorem 3.1.

Example 3.1 *Let $X = [0,1]$ and the function q: $X \times X \times \to [0, \infty)$ defined by*

$$q(x, y) = \begin{cases} x - y, \ \forall\ x \geq y, \\ 2(y - x), \ \forall\ x < y. \end{cases}$$

The pair (X, q)form an orthogonal quasi-metric space.

In the map T: $X \to X$ defined by

$$Tx = \begin{cases} \frac{x}{3}, x \in \mathbb{Q} \cap \left[0, \frac{1}{3}\right], \\ \\ 0, x \in \mathbb{Q}^c \cap \left[\frac{1}{3}, 1\right]. \end{cases}$$

Consider the set of rational numbers within the interval $X = [0, 1]$, given by:

$$X = \left\{\frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}\right\} \cap \{0\}\}.$$

By definition, two elements x and y are orthogonal if they satisfy the condition $xy < x \vee y$.
For instance, let $x = \frac{1}{5} = 0.2$ and $y = \frac{2}{5} = 0.4$. Then,

$$0.2 \times 0.4 = 0.08 < 0.2 \vee 0.4.$$

Since this inequality holds, we conclude that $x \perp y$ and $y > x$. Consequently, the quasi-metric is given by

$$q(x, y) = 2\,|\,y - x\,|.$$

Using these principles, we can compute all the quasi-metrics satisfying inequality (3.1) as follows:

$$q(Tx, T^2x) = q\left(\frac{x}{3}, \frac{x^2}{9}\right) = 2\left|\frac{x^2}{9} - \frac{x}{3}\right| = 2\left|\frac{(0.2)^2 - 3(0.2)}{9}\right| = 0.124.$$

$$q(x, Ty) = q(x, 0) = 2 \mid x - 0 \mid = 2x = 2 \times 0.2 = 0.4.$$

$$q(y, Tx) = q\left(y, \frac{x}{3}\right) = 2\left|y - \frac{x}{3}\right| = 2\left|\frac{3(0.4) - 0.2}{3}\right| = 0.67.$$

$$q(x, y) = =2 \mid y - x \mid = 2 \mid 0.4 - 0.2 \mid = 0.4.$$

$$q(x, Tx) = q\left(x, \frac{x}{3}\right) = 2\left|x - \frac{x}{3}\right| = \frac{4x}{3} = \frac{4 \times 0.2}{3} = 0.267.$$

$$q(y, Ty) = q(y, 0) = 2 \mid y - 0 \mid = 2y = 2 \times 0.4 = 0.8.$$

Now, we have two cases to verify:

(i) For $p \geq 0$ with $p = 2$, and given values $k_1 = 0.25$, $k_2 = 0.3$, $k_3 = 0.1$, $k_4 = 0.35$, we apply inequality (3.1) to obtain the following results.

$$0.124 \leq \max\left\{\min\{0.4, 0.67\}, \left[0.25(0.4)^2 + 0.3(0.267)^2 + 0.1(0.8)^2 + 0.35\left(\frac{0.4 + 0.67}{2}\right)^2\right]^{\frac{1}{2}}\right\},$$

$$0.124 \leq \max\{0.4, [0.04 + 0.2352537 + 0.064 + 0.10017875]^{\frac{1}{2}}\}.$$

$$0.124 \leq \max\{0.4, [0.43943245]^{\frac{1}{2}}\}.$$

$$0.124 \leq \max\{0.4, 0.66289\} = 0.662893.$$

(ii) For $p = 0$ and the given parameter values $k_1 = 0.25$, $k_2 = 0.3$, $k_3 = 0.1$, $k_4 = 0.35$, and $\gamma = 0.5$, we utilize inequality (3.1) to derive the following results.

$$0.124 \leq \max\{\min\{0.4, 0.67\}, 0.5[0.4]^{0.25}[0.267]^{0.3}. [0.8]^{0.1}\},$$

$$0.124 \leq \max\{\min\{0.4, 0.67\}, 0.5[0.795][0.673]. [0.977]\},$$

$$0.124 \leq \max\{0.4, 0.26\} = 0.4.$$

Thus, Theorem 3.1 is confirmed. The mapping T possesses a unique fixed point at $x = 0$ within the space X.

We provide a graphically controlled example with applications in telecommunication systems to demonstrate the results of Theorem 3.2, thereby emphasizing the originality of the established theorem.

Example 3.2 *Let the vertex set be* $X = \{0,1,4,6,8,10,12,14\}$ *and let* $q: X \times X \to [1, \infty)$ *be a mapping such that*

$$q(x, y) = 2 \mid x - y \mid.$$

The edge set $E(\Phi)$ can be defined as a function of $x \in X$ by

$$E(\Phi) = \Delta \cup \{(x, y) \in X \times X: x < y \text{ and } y - x \in \Delta(x)\},$$

where $\Delta = \{(x, x): x \in X\}$ is the diagonal, and

$$\Delta(x) = \begin{cases} \{1, 4, 6, 8, 10, 12, 14\}, & x = 0, \\ \{3, 5, 7, 9, 11, 13\}, & x = 1, \\ \{2, 4, 6, 8, 10\}, & x = 4, \\ \{2, 4, 6, 8\}, & x = 6, \\ \{2, 4, 6\}, & x = 8, \\ \{2\}, & x = 10, \\ \{2\}, & x = 12, \\ \emptyset, & x = 14. \end{cases}$$

Where Φ is an orthogonality relation and Δ-Diagonal points. Clearly, (X, q) is a graphical-controlled orthogonal quasi-metric type space (see Figure 1).

Figure 1 depicts the orthogonal relationships among signal states in a communication system, with nodes representing distinct codewords and edges indicating mutual orthogonality to prevent interference. This visualization is especially relevant for MIMO systems using space-time block coding, aiding the design of interference-free transmissions. The diagonal points show self-orthogonality, and the earlier fixed-point results ensure iterative decoding converges reliably, supporting accurate signal reconstruction under channel variations.

Let X be a set and consider a mapping $T: X \to X$ defined by

$$Tx = \begin{cases} 0, & \text{if } x \in \{0, 1, 4, 6, 8, 10, 12, 14\}, \\ 1, & \text{if } x \in \{5, 7, 9, 11, 13\}. \end{cases}$$

For the specific values $x = 0$, $y = 10$, $p \geq 0$ with $p = 2$, and constants $k_1 = 0.25$, $k_2 = 0.3$, $\mu = 0.5$, $\vartheta = 0.3$, $z = 2$, we compute the images under T as

$$Tx = T(x) = 0, \; Ty = T(y) = 0, \; T^2x = T(Tx) = 0.$$

Next, we calculate the orthogonal quasi-metrics:

$$\begin{aligned} q(x, y) &= 2\,|\,x - y\,| = 2\cdot|\,0 - 10\,| = 20, \\ q(x, Tx) &= 2\,|\,0 - 0\,| = 0, \\ q(y, Ty) &= 2\,|\,10 - 0\,| = 20, \\ q(x, Ty) &= 2\,|\,0 - 0\,| = 0, \\ q(y, Tx) &= 2\,|\,10 - 0\,| = 20, \\ q(x, T^2x) &= 2\,|\,0 - 0\,| = 0, \\ q(Tx, T^2x) &= 2\,|\,0 - 0\,| = 0. \end{aligned}$$

We then compute $M_T^z(x, y)$:

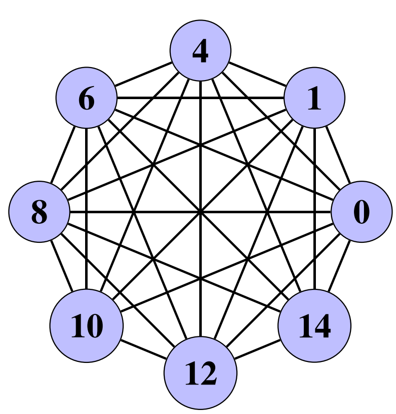


Figure 1. Orthogonal vertex set X.

$$M_T^2(x,y) = \begin{cases} \left[0.25(20)^2 + 0.3\left(\frac{0+20}{2}\right)^2\right]^{\frac{1}{2}} = 11.4, & \text{for } z \geq 0, \\ \left[\frac{0+20}{20+20}\right]^{0.25} \cdot 20^{0.3} = 2.07, & \text{for } z = 0. \end{cases}$$

For $z \geq 1$, we observe

$$0 \leq 11.4,$$

and for $z = 0$, we have

$$0 \leq 2.07.$$

Hence, the mapping T satisfies the RHR-Jaggi hybrid combinational contraction condition for the chosen points $x, y \in X$.

4 Application

This section presents several applications of the established results. Subsection 4.1 demonstrates the application of the Fixed Point Theorem to an orthogonal system within an orthogonal quasi-metric space, illustrating the findings of Theorem 3.1. Subsection 4.2 explores an application to an orthogonal system model motivated by space-time block coding in communication theory, supporting the results of Theorem 3.2. Finally, Subsection 4.3 discusses the application of Helmholtz problems with mixed boundary conditions in quasi-metric spaces, corresponding to the results established in Corollary 3.1.

4.1 Application of the fixed point theorem to an orthogonal system in an orthogonal quasi-metric space

This subsection aims to establish the connection between the results of Theorem 3.1 and orthogonal systems in telecommunication models. Recent interest has been directed toward multiple-antenna systems for their ability to support higher data rates (Jafarkhani, 2002). Foschini and Papadias (2006) proposed a quasi-orthogonal code for four transmit antennas, which, based on information-theoretic analyzes, demonstrates performance closer to the Shannon capacity compared to orthogonal codes with the same configuration. Specifically, the only orthogonal codes for four transmit antennas have rates of 1/2 and 3/4, while the suggested code achieves a rate of 1.

The structure of the quasi-system model consists of M transmit antennas and N receive antennas. A space-time block code maps an input symbol vector of length Q, denoted as $x = (x_1, x_2, \ldots, x_Q)$, into a $W \times M$ matrix $G(x)$, where W represents the block size. As a result, the code rate is $\frac{Q}{W}$.

The received signal model is given by

$$X(x) = \sqrt{\frac{\rho}{M}} G(x)H + V, \tag{4.1}$$

where H represents the $M \times N$ complex matrix X and V represents the $W \times N$ received and noise matrix. The SNR per receiver antenna is ρ. H and V are circularly symmetric complex Gaussian variables of unit variance with zero means that are independent of one another.

Consider the quasi-orthogonal code for four transmit antennas introduced in (Foschini & Papadias, 2006) which is given by

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2^* & -x_1^* & x_4^* & -x_3^* \\ x_3 & -x_4^* & -x_1 & x_2 \\ x_4^* & x_3^* & -x_2^* & -x_1^* \end{bmatrix}$$

(1) Mutual Independence: The entries of H and V are independent of each other. That means knowing one does not give any information about the other.

(2) Zero Mean: - The expected value of each entry in H and V is zero:

$$\mathbb{E}[H_{ij}] = 0, \mathbb{E}[V_{ij}] = 0$$

This implies that the random variables are centered at zero.

(3) Circularly Symmetric Complex Gaussian (CSCG): Each entry of H and V follows a complex Gaussian distribution:

$$H_{ij}, V_{ij} \sim \mathcal{CN}(0, 1)$$

(4) Unit Variance:

Each entry has a variance of 1:

$$\mathbb{E}[|H_{ij}|^2] = 1; \ [\mathbb{E}\ |V_{ij}|^2] = 1$$

This means that the real and imaginary parts of H_{ij} and V_{ij} are each Gaussian with variance $\frac{1}{2}$:

$$\Re(H_{ij}) \sim \mathcal{N}\left(0, \frac{1}{2}\right), \Im(H_{ij}) \sim \mathcal{N}\left(0, \frac{1}{2}\right)$$

MIMO Systems: When modeling wireless channels, H often represents a Rayleigh fading channel.

Random Matrix Theory: Used in spectral analysis of random matrices.

Information Theory: Analysis of Gaussian noise in complex-valued systems.

Next, we established the following theorem:

Theorem 4.1 *Suppose the conditions below holds:*

(a) Let $G(x)$ be a continuous $W \times M$ matrix in X.

(b) There exists two constant $\Delta, \gamma \in [0, 1]$ such that

$$|\ G(x) - G(y)| = 2\Delta\ |\ x - y\ |,$$

and

$$|\ G(x) - G(y)| = 2\gamma\ |\ x - y\ |,$$

with

$$\Delta \vee \gamma = \sqrt{\frac{\rho}{M}}H \leq 1.$$

(c) For $p \geq 0$ and $p = 0$ we have

(i) $q(x, y) = 2\ |\ x - y\ |=$

$$M_T^p(x, y) = \begin{cases} \left[k_1(q(x, y))^p + k_2(q(x, Tx))^p + k_3(q(y, Ty))^p \ + k_4\left(\frac{q(x, Ty) + q(y, Tx)}{2}\right)^p\right]^{\frac{1}{p}}, \mathit{for}\ \ p \\ \geq 0, Z, K \in X; \end{cases}$$

(ii) $q(x, y) = 2\ |\ x - y\ |=$

$$M_T^p(x, y) = \begin{cases} \gamma\,[q(x, y)]^{k_1}[q(x, Tx)]^{k_2}.\ [q(y, Ty)]^{k_3}, \\ \mathit{for}\ \ p = 0, k_4 = 0x, y \in X. \end{cases}$$

(iii) $|\ TX(x) - TY(y)| = |\ Tx - T^2x\ |.$

Then T has a unique fixed point.

Proof. Consider a mapping $T: X \to X$ defined by

$$TX(x) = \sqrt{\frac{\rho}{M}}G(x)H + V.$$

We show that T is a contraction. we claim that $TX \neq TY$, for $x \in TX$ and $K \in TY$, we have

$$|TX(x) - TY(y)| \leq \left|\sqrt{\frac{\rho}{M}}G(x)H + V - \sqrt{\frac{\rho}{M}}G(y)H - V\right|,$$

$$|TX(x) - TY(y)| \leq \sqrt{\frac{\rho}{M}}H\,|G(x) - G(y)|,$$

$$|TX(x) - TY(y)| \leq \sqrt{\frac{\rho}{M}}H\,|G(x) - G(y)|,$$

$$|TX(x) - TY(y)| \leq 2\sqrt{\frac{\rho}{M}}H\,|x - y|.$$

In conclusion, we have the following observations:
(i) For $p \geq 0$, we get

$$q(Tx, T^2x) \leq \Delta\left[k_1(q(x,y))^p + k_2(q(x,Tx))^p + k_3(q(y,Ty))^p + k_4\left(\frac{q(x,Ty) + q(y,Tx)}{2}\right)^p\right]^{\frac{1}{p}},$$

(ii) For $p = 0$, we obtain

$$q(Tx, T^2x) \leq \gamma\,[q(x,y)]^{k_1}\,[q(x,Tx)]^{k_2}.\,[q(y,Ty)]^{k_3}.$$

Hence, equation (4.1) has a unique transmit antenna. Thus, all conditions imposed in Theorem 3.1 and Theorem 4.1 are satisfied. □

4.2 Application of orthogonal systems based on spacetime block coding in communication theory

In this subsection, we consider an orthogonal system model inspired by spacetime block coding in communication theory (Jafarkhani, 2002; Tarokh et al., 1998). Let $\mathscr{H}$ be the signal space and $\{e_i\}_{i=1}^n$ an orthogonal basis representing independent transmission channels. The channel between transmit antenna and the receive antenna zero is denoted by h_0 and between the transmit antenna and the receive antenna one is denoted by h_1 where

$$h_0 = \alpha_0 e^{j\theta_0},$$

$$h_1 = \alpha_1 e^{j\theta_1},$$

The encoder/iteration is modeled by a nonlinear transformation $T: \mathscr{H} \to \mathscr{H}$ defined as

$$Tx = \sum_{i=0}^{n} h_i\langle x, e_i\rangle + f(x), \tag{4.2}$$

where α_i are channel-dependent attenuation coefficients, and $f(x)$ accounts for nonlinear perturbations due to noise or interference.

Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\{e_1, e_2, \ldots, e_n\}$ be the standard orthonormal basis. Then

$$\langle x, e_i \rangle = \sum_{j=1}^{n} x_j e_{i,j}.$$

Since

$$e_{i,j} = \begin{cases} 1, & j = i, \\ 0, & j \neq i, \end{cases}$$

it follows that

$$\langle x, e_i \rangle = x_i.$$

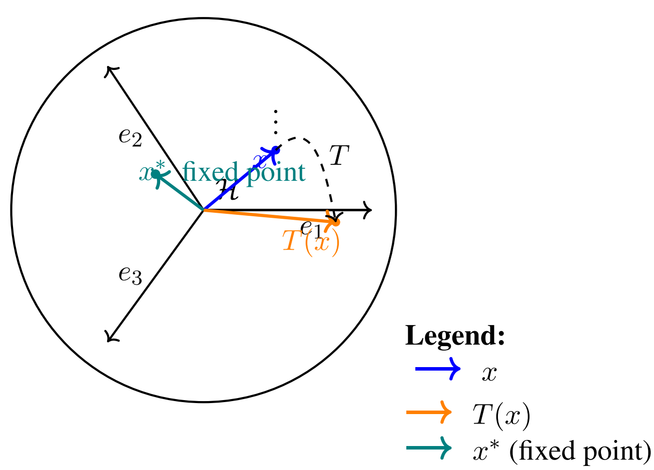
Equation (4.2) is equivalent to

$$Tx = \sum_{i=0}^{n} h_i x_i + f(x), \tag{4.3}$$

Consequently, we have

$$Tx = \sum_{i=0}^{n} \alpha_i e^{j\theta_i} x_i + f(x). \tag{4.4}$$

We now establish the existence of a solution for the orthogonal system model inspired by spacetime block coding in communication theory (4.4).



Theorem 4.2 *Let* $L: [0,1] \times \mathbb{R} \to \mathbb{R}$ *and* $q: \mathbb{R} \to \mathbb{R}$ *be mappings satisfying the following conditions:*

(i) For all $t \in [0, 1]$ and $x, y \in [0, 1]$ with $q(x, y) \geq 0$,

$$|L(t, x) - L(t, y)| \leq \Delta \sup_{x,y \in [0,1]} 2|x - y|.$$

where

$$q(x, y) = |x - y| \leq \mu q(x, Tx) + \vartheta q(x, T^2x) + M_T^z(x, y), \forall x \in X, \tag{4.5}$$

and $\Delta = \sum_{i=0}^{n} \alpha_i e^{j\theta_i} \leq 1$.

(ii) For all $x, y \in [0, 1] = X$,

$$q(Tx, T^2x) \geq 0,$$

where $T: X \to X$ is a given mapping.

Then, the nonlinear transformation equation (4.4) has at most one solution in X. □

Proof. Let $X = C([0, 1], \mathbb{R})$ denote the Banach space of continuous real-valued functions on $[0, 1]$ equipped with the quasi-uniform norm

$$q(x, y) := \|x - y\|_\infty = \sup_{t \in [0,1]} |x(t) - y(t)|.$$

Define the operator $T: X \to X$ by

$$Tx := \sum_{i=0}^{n} \alpha_i e^{j\theta_i} x_i(t) + f(x), t \in [0, 1].$$

By hypothesis (i), for all $t \in [0, 1]$ and $x, y \in X$ we have

$$\begin{aligned} \|Tx - Ty\|_\infty &\leq \|\textstyle\sum_{i=0}^{n} \alpha_i e^{j\theta_i} x_i(t) + f(x) - \sum_{i=0}^{n} \alpha_i e^{j\theta_i} y_i(t) + f(y)\|_\infty, \\ \|Tx - Ty\|_\infty &\leq \textstyle\sum_{i=0}^{n} \alpha_i e^{j\theta_i} \|x_i(t) - y_i(t)\|_\infty. \end{aligned} \tag{4.6}$$

Taking the supremum over $t \in [0, 1]$ gives

$$\|Tx - Ty\|_\infty \leq 2\Delta\|x - y\|_\infty.$$

$$q(Tx, Ty) \leq \Delta q(x, y).$$

$$q(Tx, Ty) \leq \mu q(x, Tx) + \vartheta q(x, T^2x) + M_T^z(x, y).$$

Thus, T is a Lipschitz mapping with constant Δ.

If $\Delta < 1$, then T is a contraction on the complete quasi-metric space (X, q). By the Banach contraction mapping principle, T admits a unique fixed point $x^* \in X$, i.e.

$$x^*(t) = Tx^*(t), t \in [0, 1].$$

Therefore, the nonlinear transformation Equation (4.4) has exactly one solution in X. □

The communication system achieves *stability* if there exists a signal $x^* \in \mathscr{H}$ such that $T(x^*) = x^*$. In our framework, this condition corresponds to a *fixed point* of T. By employing the Rus-Hicks-Rhoades type contraction in the orthogonal quasi-metric space $(\mathscr{H}, q)$, we show that such a fixed point exists and is unique under admissible parameter ranges. Concretely, the orthogonality ensures that cross-channel interference terms vanish, while the quasi-metric q captures directional asymmetries between forward and backward channel operations, a property not handled in classical metric settings.

4.3 Application of Helmholtz problems with mixed boundary conditions in quasi-metric spaces

This subsection discusses the widespread use of fixed-point theory in various abstract spaces and extends its application to Helmholtz problems with mixed boundary conditions in orthogonal quasi-metric spaces, illustrating an application of the results obtained in Corollary 3.1. The Helmholtz problem with mixed boundary conditions arises when solving the Helmholtz equation

$$-\Delta u - \lambda u = f \ \text{ in } \ \Omega, \tag{4.7}$$

on a domain Ω, subject to different types of boundary conditions on different portions of the boundary $\partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\lambda > 0$ is a spectral parameter, u is the unknown function to be solved and $f : \Omega \to \mathbb{R}$ is a source function. Specifically:

(i) Dirichlet condition $u = 0$ is prescribed on part of the boundary Γ_D,
(ii) Neumann condition $\frac{\partial u}{\partial n} = 0$ on another part Γ_N, and
(iii) Robin condition $\frac{\partial u}{\partial n} + \alpha u = 0$ on the remaining part Γ_R.

Such problems naturally appear in acoustics, electromagnetics, and quantum mechanics, where the wave behavior is influenced by how the boundary interacts with the field. Mixed conditions model real-world situations like waveguides, scattering surfaces, and resonators, where some boundaries absorb or reflect waves differently. Mathematically, the mixed boundary Helmholtz problem is well-posed in suitable function spaces, and its solutions are linked to eigenvalue problems. The eigenfunctions form an orthogonal system, and the Green's function or variational methods are often used to represent solutions.

Consider the Helmholtz problem with mixed boundary conditions (Younis et al., 2025) also provides a framework for studying vibrations in physical and engineering contexts. For instance, the condition

$$u''(s) + \lambda u(s) = f(s),\ u(0) = 0,\ u'(1) = 0. \tag{4.8}$$

captures standing wave patterns within the interval [0, 1]. When the eigenvalue $\lambda = \pi^2/4$, the system corresponds to a fundamental resonant mode characterized by half-wavelength fitting the domain. This setup is widely applicable: in acoustics, it models air column vibrations in closed-open tubes; in electromagnetics, it reflects field distributions in waveguides and resonant cavities; and in mechanics, it represents vibrations in beams with one fixed and one free end. Such formulations demonstrate how Helmholtz phenomena under mixed BVPs inform wave propagation, resonance control, and stability in practical systems, thereby enriching both the theoretical landscape of fixed-point analysis and its engineering applications.

The problem (4.8) has a solution x in the space $X = C([0, 1], \mathbb{R})$, which is represented by the integral equation

$$x(t) = \int_0^1 \dot{G}(t, s) f(s, t, x(t))\, ds,\ t \in [0, 1]. \tag{4.9}$$

where $\dot{G}(t, s)$ is the Green's function for any λ, $s > 0$, given by

$$\dot{G}(t, s) = \begin{cases} \frac{\sin(\sqrt{\lambda}\, t)}{\sqrt{\lambda}\sin(\sqrt{\lambda})} \sin(\sqrt{\lambda}(1 - s)), & 0 \le t \le s, \\ \frac{\sin(\sqrt{\lambda}\, s)}{\sqrt{\lambda}\sin(\sqrt{\lambda})} \sin(\sqrt{\lambda}(1 - t)), & s \le t \le 1. \end{cases}$$

The integral of $\dot{G}(t, s)$ over [0, 1] is

$$\begin{aligned} \int_0^1 \dot{G}(t, s)\, ds \ &= \frac{1}{\lambda \sin(\sqrt{\lambda})}[\sin(\sqrt{\lambda}(1 - t))(1 - \cos(\sqrt{\lambda}\, t)) \\ &\quad + \sin(\sqrt{\lambda}\, t)(1 - \cos(\sqrt{\lambda}(1 - t)))]. \end{aligned}$$

The admissible eigenvalues are

$$k = \frac{(2n + 1)\pi}{2},\ \lambda_n = k^2 = \left(\frac{(2n + 1)\pi}{2}\right)^2,\ n = 0, 1, 2, \ldots$$

with corresponding eigenvalue functions

$$u_n(s) = \sin\left(\frac{(2n+1)\pi}{2}s\right).$$

For $n = 2$ the choice $\lambda = \frac{9\pi^2}{4}$ and $t = \frac{1}{2}$ we have

$$\sqrt{\lambda} = \frac{3\pi}{2},\ 1 - t = \frac{1}{2}.$$

Hence

$$\int_0^1 \dot{G}\left(\frac{1}{2}, s\right)ds = \frac{1}{\lambda \sin(\sqrt{\lambda})}[\sin(\sqrt{\lambda}(1-t))(1 - \cos(\sqrt{\lambda}t)) + \sin(\sqrt{\lambda}t)(1 - \cos(\sqrt{\lambda}(1-t)))].$$

Since $\sqrt{\lambda}t = \sqrt{\lambda}(1-t) = \frac{3\pi}{4}$, this reduces to

$$\int_0^1 \dot{G}\left(\frac{1}{2}, s\right)ds = \frac{2}{\lambda \sin(\sqrt{\lambda})}\sin\left(\frac{3\pi}{4}\right)(1 - \cos\left(\frac{3\pi}{4}\right)).$$

Now,

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2},\quad \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},$$

so that

$$\sin\left(\frac{3\pi}{4}\right)\left(1 - \cos\left(\frac{3\pi}{4}\right)\right) = \frac{\sqrt{2}}{2}\left(1 + \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}+1}{2}.$$

Therefore,

$$\int_0^1 \dot{G}\left(\frac{1}{2}, s\right)ds = \frac{\sqrt{2}+1}{\lambda \sin(\sqrt{\lambda})}.$$

Finally, since $\sin(\sqrt{\lambda}) = \sin\left(\frac{3\pi}{2}\right) = -1$ and $\lambda = \frac{9\pi^2}{4}$, we obtain

$$\int_0^1 \dot{G}\left(\frac{1}{2}, s\right)ds = -\frac{4(\sqrt{2}+1)}{9\pi^2}.$$

Now, we introduce the following theorem.

Theorem 4.3 *Suppose the following conditions are satisfied:*

(i) For every $t \in [0, 1]$ and for all $x, s \in X$, the function f satisfies

$$|f(s, t, x(t)) - f(s, t, y(t))| \leq L\,|x(t) - y(t)|.$$

where

$$q(x, y) = 2\,|x(t) - y(t)| = \psi(\tau \mathscr{M}_q(x, y)),$$

where

$$\mathscr{M}_q(x, y) = \max\{\min\{q(x, y), q(x, Ty), q(y, Tx)\}, M_T^s(x, y)\}.$$

(ii)

$$\int_0^1 \dot{G}\left(\frac{1}{2}, s\right)ds = -\frac{4(\sqrt{2}+1)}{9\pi^2} \leq 1.$$

Then, any solution to the integral equation given in (4.9) also provides a solution to the Helmholtz problem stated in (4.8).

Proof. Define an operator T: $X \to X$ by

$$(Tx)(t) = \int_0^1 \dot{G}(t, s)f(s, t, x(t))ds, \; t \in [0, 1].$$

We aim to show that T has a unique fixed point $x^* \in X$, i.e. $Tx^* = x^*$, which will correspond to a solution of the integral equation.

By assumption (i), for all $x, y \in X$ and $t \in [0, 1]$, we have an estimate of the operator difference.

$$\|Tx - Ty\| = \sup_{t\in[0,1]} \left| \int_0^1 \dot{G}(t, s)(f(s, t, x(t)) - f(s, t, y(t)))ds \right|$$

$$\leq \sup_{t\in[0,1]} \int_0^1 |\dot{G}(t, s)| \, |f(s, t, x(t)) - f(s, t, y(t))| \, ds$$

$$\leq 2 \sup_{t\in[0,1]} \int_0^1 |\dot{G}(t, s)| \, |x(t) - y(t)| \, ds$$

$$= q(x, y)\cdot \sup_{t\in[0,1]} \int_0^1 |\dot{G}(t, s)| \, ds.$$

$$= q(x, y)\left| -\frac{4(\sqrt{2}+1)}{9\pi^2} \right|,$$

$$\|Tx - Ty\| \leq Lq(x, y).$$

If $L < 1$, T is a contraction with respect to the q-metric induced by ψ and $\mathscr{M}_q$. Then

$$q(x, y) = 2\,|\,x(t) - y(t)\,| = \psi(\tau \mathscr{M}_q(x, y)),$$

where

$$\mathscr{M}_q(x, y) = \max\{\min\{q(x, y), q(x, Ty), q(y, Tx)\}, M_T^s(x, y)\}.$$

Since T is a contraction on the complete orthogonal quasi-metric space $(X, \|\cdot\|)$, there exists a unique $x^* \in X$ such that

$$Tx^* = x^*.$$

Hence, the integral equation

$$x(t) = \int_0^1 \dot{G}(t, s)f(s, t, x(t))ds$$

has a unique solution x^* in X, completing the proof. □

5 Conclusion

In this study, we have developed new fixed point theorems for generalized combinational hybrid-type contraction mappings within the context of orthogonal quasi-metric spaces. These theorems broaden and generalize classical fixed point results by introducing wider classes of contraction mappings, including: Rus-Hicks-Rhoades-Cirić hybrid combinational contractions, Orthogonal *RHR*-Jaggi hybrid combinational type mappings, and ψ-orthogonal Reich-Rus-Cirić hybrid combinational type contractions. To demonstrate the applicability and strength of our theoretical results, we provided relevant examples, including applications to orthogonal system models. Specifically, our work builds upon and extends the previous research by Gordji et al. (2017), Jleli and Samet (2012), Karapinar and Fulga (2019, 2022), and Shehu et al. (2025).

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