Fixed point theorems for extended interpolative Kanann-Ćirić-Reich-Rus non-self type mapping in hyperbolic complex-valued metric space

Lucas Wangwe[1], Laxmi Rathour[2,*], Lakshmi Narayan Mishra[3] and Vishnu Narayan Mishra[4]

[1]Department of Mathematics, College of Science and Technical Education, Mbeya University of Science and Technology, Tanzania
[2]Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India
[3]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
[4]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur 484 887, Madhya Pradesh, India
* Corresponding author

E-mail: wangwelucas@gmail.com[1], laxmirathour817@gmail.com[2], lakshminarayanmishra04@gmail.com[3], vnm@igntu.ac.in[4]

Abstract

This paper aims to demonstrate the fixed point theorem for extended interpolative non-self-type contraction mapping in hyperbolic complex-valued metric spaces. We provide an example for verification of the results. Further, as an application, we prove the existence and uniqueness of solutions for a class of Hadamard partial fractional integral equations by applying some fixed point theorems.

2020 Mathematics Subject Classification. **47H10**. 54H25.
Keywords. fixed point, extended interpolative type mapping, hyperbolic space, complex-valued metric space, Hadamard fractional integral equation.

1 Introduction

The study of non-linear contraction mapping was initiated by Boyd and Wong [5] in 1969. Then, followed by Carmargo [10] showed an application of a fixed point theorem of Boyd and Wong [5] in metric spaces. Assad and Kirk [2] formulated fixed point theorems for non-self mappings in metrically convex metric spaces. Their result was extended by Rhoades [31] who proved the existence of fixed points in metric spaces for what he termed as generalized contractions, which set out sufficient conditions for a non-self mapping to have a unique fixed point in a metric space. Imdad and Kumar [16] extended the theorem by Rhoades by demonstrating the existence of common fixed points for pairs of non-self mappings obeying specified conditions in a metrically convex metric space. Kirk [19] gave the Krasnoselskii's iteration process in hyperbolic space. Berinde *et al.* [4] proved a fixed point theorems for non-self single-valued almost contractions. Reich and Shafrir [30] proved nonexpansive iterations in hyperbolic spaces. Shafrir [34] proved the approximated fixed point property in Banach and hyperbolic spaces. Ćirić *et al.* [11] proved a common fixed point theorems for non-self-mappings in metric spaces of hyperbolic type. Followed by Eke *et al.* [13] proved the common fixed point theorems for non-self mappings of nonlinear contractive maps in convex metric spaces. Recently, Wangwe and Kumar [42] proved the fixed point theorem for multivalued non-self mappings in Partial Symmetric Spaces.

Advanced Studies: Euro-Tbilisi Mathematical Journal 17(2) (2024), pp. 1–21.
DOI: 10.32513/asetmj/1932200824017
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 25 June 2023.
Accepted for publication: 30 Juner 2023.

In 2011, Azam *et al.* [3] extended the metric space to complex-valued metric spaces and established the existence of fixed point theorems under the contraction condition in rational expression. After that, several works were discovered. Rouzkard and Imdad [32] established some common fixed point theorems on complex-valued metric spaces. Sintunavarat and Kumam [36] gave a generalized common fixed point theorems in complex-valued metric spaces and applications. Sintunavarat *et al.* [37] introduced Urysohn integral equations approach by common fixed points in complex-valued metric spaces. Klin-eam and Suanoom [20] gave some common fixed point theorems for generalized contractive type mappings on complex-valued metric spaces. Kumam *et al.* [23] gave the fixed point results satisfying rational type contractive conditions in complex-valued metric spaces.

Karapinar [17] gave a novelty on interpolative Kannan mapping in metric spaces. Followed by Mishra *et al.* [24] proved the interpolative Reich-Rus-Ćirić and Hardy-Rogers contraction on quasi-partial b-metric space and related fixed point results. Gautam *et al.* [14] proved Interpolative Chatterjea and cyclic Chatterjea contraction on quasi-partial b-metric space. Wangwe and Kumar [41] proved fixed point results for interpolative Ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric space with applications. Wangwe [39] proved fixed point theorems for interpolative Kanann contraction mappings in Busemann space with an application to a matrix equation, and Wangwe [40] proved fixed point theorem for interpolative mappings in F-M_v-metric space with an application.

The innovation of this paper is to generalize the results of Mohammadi *et al.* [25], Karapinar [17], Dass and Gupta [12] and Eke *et al.* [13] and other related works to proved the fixed point theorems for extended interpolative Kanann-Ćirić-Reich-Rus non-self non-linear contractive mapping in hyperbolic complex metric spaces. We provide an example for verification of the results. Further, as an application, we prove the existence and uniqueness of solutions for a class of Hadamard partial fractional integral equations by applying some fixed point theorems.

2 Preliminaries

In this section, we begin with some definitions and theorems for enhancing the main results.

Definition 2.1. [21] Let (X, d, W) be a hyperbolic space if (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is a function satisfying

(i) $\forall\ u, v \in X$ and $\forall \lambda \in [0,1]$.

$$d(z, W(u, v, \lambda)) \leq (1-\lambda)(z, u) + \lambda d(z, v),$$

(ii) $\forall\ u, v \in X$ and $\forall \lambda_1, \lambda_2 \in [0,1]$.

$$d(W(u, v, \lambda_1), W(u, v, \lambda_2)) \leq |\lambda_1 - \lambda_2| d(u, v),$$

(iii) $\forall\ u, v \in X$ and $\forall \lambda \in [0,1]$.

$$W(u, v, \lambda) = W(v, u, 1-\lambda),$$

(iv) $\forall\ u, v, z, w \in X$ and $\forall \lambda \in [0,1]$.

$$d(W(u, z, \lambda), W(v, w, \lambda)) \leq (1-\lambda) d(u, v) + \lambda d(z, w).$$

If only condition (i) is satisfied, then (X, d, W) is a convex metric space by Takahashi concepts [38]. According to Shafrir [34], suppose that (X, d) is a complete metric space. Assume that there is a family W of metric lines (isometric images of $\mathbb{R}$) in X such that for every $u, v \in X$, $u \neq v$, there is z unique line in W that passes through u and v. The closed metric segment connecting u and v will be denoted by $[u, v]$.

Definition 2.2. [27] Let X be a vector space. A subset $K \subset X$ is said to be affinely convex if, for all $u, v \in K$, the affine segment

$$[u, v] := \Big\{(1-\lambda)u + \lambda v : \lambda \in [0, 1]\Big\}$$

is contained in X.

The unique point $z \in [u, v]$ satisfying

$$d(u, z) = \lambda d(u, v)$$

and

$$d(z, v) = (1-\lambda)d(u, v).$$

Therefore, (X, d, W), is a hyperbolic space if

$$d\left(\frac{1}{2}u \oplus \frac{1}{2}v, \frac{1}{2}u \oplus \frac{1}{2}z\right) \quad \leq \quad \frac{1}{2}d(v, z).$$

Equivalent to

$$d((1-\lambda)u \oplus \lambda z, (1-\lambda)v \oplus \lambda w) \quad \leq \quad (1-\lambda)d(x, y) + \lambda d(z, w).$$

Consequently, we have

$$d(W(u, z, \lambda), W(v, w, \lambda)) \quad \leq \quad (1-\lambda)d(u, v) + \lambda d(z, w).$$

Then a point $z \in [u, v]$ if and only if there exists $\lambda \in [0, 1]$ such that $d(z, u) = \lambda d(u, v)$ and $d(z, v) = (1-\lambda)d(u, v)$. For simplicity, we will write

$$z = (1-\lambda)x \oplus \lambda y.$$

If X is a hyperbolic space the metric d on X is convex. This means that for any $z \in X$,

$$d(z, (1-\lambda)u \oplus \lambda y) \leq (1-\lambda)d(z, u) + \lambda d(z, v), \tag{2.1}$$

for all $t \in [0, 1]$.

Lemma 2.1. [26] Let (X, d, W) be a convex metric space, then the following is true:

$$d(u, v) \quad = \quad d(u, W(u, v, \lambda)) + d(v, W(u, v, \lambda)) \tag{2.2}$$

for all $(u, v, \lambda) \in X \times X \times I$.

$$d\Big(u, W\Big(u, v, \frac{1}{2}\Big)\Big) = d\Big(v, W\Big(u, v, \frac{1}{2}\Big)\Big) \quad = \quad \frac{1}{2}d(u, v) \tag{2.3}$$

for all $u, v \in X$.

An example of hyperbolic space which is a CAT(0) space.

Example 2.1. [7, 29] A Hadamard space is a non-empty complete metric space (X, d) with the property that for any pair of points $u, v \in X$, there exists a point $m \in X$ such that

$$d(z,m)^2 + \frac{d(u,v)^2}{4} \quad \leq \quad \frac{d(z,u)^2 + d(z,v)^2}{2}, \tag{2.4}$$

Azam *et al.* [3] gave the following concept on complex-valued metric spaces.

Definition 2.3. Let $\mathbb{C}$ be a set of complex number and $w_1, w_2 \in \mathbb{C}$ define a partial order $\preceq$ on $\mathbb{C}$ as follows: $w_1 \preceq w_2$ if and only if one of the following conditions is satisfied:

(C1) $\Re(w_1) = \Re(w_2), \Im(w_1) < \Im(w_2)$,

(C2) $\Re(w_1) < \Re(w_2), \Im(w_1) = \Im(w_2)$,

(C3) $\Re(w_1) < \Re(w_2), \Im(w_1) < \Im(w_2)$,

(C4) $\Re(w_1) = \Re(w_2), \Im(w_1) = \Im(w_2)$.

In particular, we will write $w_1 \preceq w_2$ if $w_1 \neq w_2$ and all of $(C1), (C3)$ and $(C4)$ is satisfied and we will write $w_1 \preceq w_2$ if only $(C4)$ is satisfied.

Also, it is known that

(i) If $0 \preceq w_1 \precnsim w_2$, then $|w_1| < |w_2|$.

(ii) If $w_1 \preceq w_2$ and $w_2 \preceq w_3$, then $w_1 \prec w_3$.

The following is the distance function in complex-valued metric space:

Definition 2.4. *[3]* Let X be a non-empty set. Assume that a mapping $d : X \times X \to \mathbb{C}$ is a distance on a complex-valued metric space if the following metric hold:

(CM1) $0 \preceq d(u, v)$ for all $u, v \in X$ and $d(u, v) = 0 \Longleftrightarrow u = v$,

(CM2) $d(u, v) = d(v, u)$ for all $u, v \in X$,

(CM3) $d(u, v) \preceq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Then d is called a complex-valued metric on X and (X, d) is called a complex-valued metric space.

The following are examples which satisfy the axioms of complex-valued metric space.

Example 2.2. [35] Let $X = \mathbb{C}$ be a se of complex number, define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. By

$$d(w_1, w_2) = |u_1 - u_2| + i|v_1 - v_2|,$$

where $w_1 = u_1 + iv_1$ and $w_2 = u_2 + iv_2$. Then $(\mathbb{C}, d)$ is a complex-valued metric space.

Example 2.3. [36] Let $X = \mathbb{C}$. Define a metric $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$d(w_1, w_2) = |w_1 - w_2|e^{ik},$$

where $k \in [0, \frac{\pi}{2}]$. Then $(\mathbb{C}, d)$ is a complex-valued metric space.

The Cauchy sequence convergence properties on completeness of complex-valued metric spaces.

Definition 2.5. [3] Suppose that (X, d) is a complex-valued metric space.

(i) A sequence $\{u_n\}$ converges to an element $u \in X$ if for every $0 \prec t \in \mathbb{C}$ there exist an integer $\mathbb{N}$ such that

$$d(u_n, u) \prec t,$$

for all $n \geq \mathbb{N}$. We write this by

$$\lim_{n\to\infty} d(u_n, u) \implies u_n \to u \ as \ n \to \infty.$$

(ii) If for any $t \in \mathbb{C}$ with $0 \prec t$, there exist $N \in \mathbb{N}$ such that, for all $n > N$,

$$d(u_n, u_{n+m}) \prec t,$$

where $m \in \mathbb{N}$, then $\{u_n\}$ is called a Cauchy sequence in X.

(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex-valued metric space.

Lemma 2.2. *[3]* Let (X, d) be a complex-valued metric space and (x_n) be a sequence in X. Then $\{u_n\}$ converges to a point $u \in X$ if and only if

$$|d(u_n, u)| \to 0 \ as \ n \ \to \infty.$$

Lemma 2.3. *[3]* Let (X, d) be a complex-valued metric space and (u_n) be a sequence in X. Then $\{u_n\}$ is a Cauchy sequence if and only if

$$|d(u_n, u_{n+m})| \to 0 \ as \ n \ \to \infty.$$

Definition 2.6. [6] Let (X, d) be a metric space. A mapping $\mathrm{T} : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{v_n\}$, if $\{\mathrm{T}v_n\}$ is convergence then $\{v_n\}$ also is convergence. T is said to be subsequentially convergent if we have, for every sequence $\{v_n\}$, if $\{v_n\}$ is convergence then $\{v_n\}$ has a convergent subsequence.

The concept of interpolative mapping in Banach space was introduced by Krein *et al.* [22] in the following way:

Definition 2.7. Let (A, B) and (C, D) be two Banach couples. A linear mapping T acting from the space $A + B$ to $C + D$ is called a bounded operator from (A, B) to (C, D) if the restrictions of T to the space A and B are bounded operator from A to C and B to D, respectively. It is denoted by $\mathrm{L}(AB, CD)$ the linear space of all bounded operators from the couple (A, B) to the couple (C, D). This Banach space in the norm.

$$\|\mathrm{T}\|_{\mathrm{L}(AB,CD)} = \max\{\|\mathrm{T}\|_{A\to B}, \|\mathrm{T}\|_{C\to D}\}. \tag{2.5}$$

If the operator T_n from a Cauchy sequence in $\mathrm{L}(AB, CD)$, then their restriction to A and C converge in $\mathrm{L}(A, C)$ and $\mathrm{L}(B, D)$ to operator T' and T'' which coincides on $A \cap B$ then the sequence T_n converges in $\mathrm{L}(AB, CD)$ to uniquely defined operator T' acting from $A + B$ to $C + D$ which implies that

$$\mathrm{T}(u, v) \quad = \quad \mathrm{T}'u + \mathrm{T}''v, \tag{2.6}$$

$u \in A$ and $v \in B$.

Definition 2.8. [22] The triple (A, B, E) is said to be an interpolation triple of type $\gamma(0 \leq \alpha \leq 1)$ relative to (C, D, F) if it's an interpolation triple and the following inequality holds:

$$\|\mathrm{T}\|_{E \to F} \quad = \quad c\|\mathrm{T}\|^{\gamma}_{A \to B} \cdot \|\mathrm{T}\|^{1-\gamma}_{C \to D}, \tag{2.7}$$

for some constant c.

In case where A coincides with C, B with D, and E with F, E is said to be an interpolation space of type γ between A and B.

The following results for interpolative Kannan contraction have been proved in [17] as follows:

Definition 2.9. [17] Let (X, d) be a metric space, a mapping $\mathrm{T} : X \to X$ is said to be interpolative Kannan contraction mappings if

$$d(\mathrm{T}u, \mathrm{T}v) \quad \leq \quad \vartheta[d(u, \mathrm{T}u)]^{\delta}.[d(v, vv)]^{1-\delta}, \tag{2.8}$$

for all $u, v \in X$ with $u \neq \mathrm{T}u$, where $\vartheta \in [0, 1)$ and $\delta \in (0, 1)$.

Theorem 2.1. [17] Let (X, d) be a complete metric space and T be an interpolative Kannan-type contraction. Then T has a unique fixed point in X.

Mohammadi *et al.* [25] gave the following definition and theorem for the extended interpolative Ćirić-Reich-Rus type F-contraction mappings in metric space.

Definition 2.10. [25] Let (X, d) be a metric space, we say that the self-mapping $\mathrm{T} : X \to X$ is an extended interpolative Ćirić-Reich-Rus type F-contraction mappings if there exists $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(\mathrm{T}u, \mathrm{T}v)) \quad \leq \quad \alpha F(d(u, v)) + \beta F(d(u, \mathrm{T}u)) + (1 - \alpha - \beta)F(d(v, \mathrm{T}v)),$$

for all $u, v \in X \backslash Fix(\mathrm{T})$ with $u \neq \mathrm{T}u$ with $d(\mathrm{T}u, \mathrm{T}v) > 0$.

Theorem 2.2. [25] Let (X, d) be a complete metric space and T be an interpolative Ćirić-Reich-Rus type F-contraction. Then T admits a fixed point in X.

Beride and Pacurar [4] gave the following definition and theorem for almost non-self contraction mapping using M property given as follows:

Definition 2.11. [4] Let X be a hyperbolic space, K a non-empty closed subset of X and $\mathrm{T} : K \to X$ be an extended interpolative non-self mapping. Let $u \in K$ with $\mathrm{T}u \notin K$ and let $v \in \partial K$, such that

$$d(v, \mathrm{T}v) \quad \leq \quad d(u, \mathrm{T}u),$$

where

$$d(u, \mathrm{T}u) \quad \leq \quad d(u, v) + d(v, \mathrm{T}u),$$

for the corresponding $v \in Y$.

Theorem 2.3. *[4]* Let (X, d) be a Banach space, K a nonempty closed subset of X and $\mathrm{T} : K \longrightarrow X$ a nonself almost contraction if there exists a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(\mathrm{T}u, \mathrm{T}v) \quad \leq \quad \delta d(u, v) + Ld(v, \mathrm{T}u), \forall u, v \in X.$$

If T satisfies Rother's boundary condition $\mathrm{T}(\partial K) \subset K$. Then T has a fixed point in K.

3 Main results

In this section, we prove the main results as follows:

Theorem 3.1. Let (X, d) be a convex hyperbolic complex metric space, K a non-empty closed subset of X, and ∂K the boundary of K. Let ∂K be non empty and and $\mathrm{T} : K \to X$ be an extended interpolative Kanann-Ćirić-Reich-Rus type contraction mappings of K into X. If there exists three constants α, β and ϑ with $\alpha + \beta + \vartheta < 1$ satisfy the following conditions:

(i) $\mathrm{T}(\partial K) \subset K$,

(ii) T has property M,

(iii) T is sequentially convergent in K,

(iv)

$$d(\mathrm{T}u, \mathrm{T}v) \quad \preceq \quad \vartheta[\alpha d(u, v) + \beta d(u, \mathrm{T}u)] + (1 - \alpha - \beta)d(v, \mathrm{T}v), \tag{3.1}$$

for all $u, v \in K$.

(v) if $z \in$ affine convex segment $[u, v]$ and $m = \frac{u+v}{2}$ the midpoint of $[u, v]$, then the Hadamard extended the interpolative equation

$$\Big[\frac{d(u, v)}{2}\Big]^2 \quad \preceq \quad \alpha\Big[\frac{d(z, u)}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(z, v)}{\sqrt{2}}\Big]^2 - (1 - \alpha - \beta)\Big[d(z, m)\Big]^2, \tag{3.2}$$

is satisfied with $\triangle(z, u, v)$ a geodesic triangle in K and $0 < \vartheta \leq 1$, $\vartheta = \sqrt{\frac{2\alpha+4\beta-4}{4-\beta}}$.

Then T admits the unique fixed point in $K \in X$.

Proof. Let x_0 be an arbitrary point in X. If $\mathrm{T}(K) \subset K$, T is a self-extended interpolative mapping on a closed set K and $X = K$. Otherwise, we consider the case $\mathrm{T}(K) \nsubseteq K$, $X \neq K$ and $\mathrm{T} : K \to X$ a non-self extended interpolative mapping. Let $u_0 \in \partial K$. Using condition (i) of Theorem 3.1 we have $\mathrm{T}u_0 \in K$. Denotes by $u_1 = \mathrm{T}u_0$ that is $u_0 \in K$. If $\mathrm{T}u_1 \in K$, implies that $u_2 \in K$

and $u_2 = \mathrm{T}u_1$. Our proof is completed. If $\mathrm{T}u_1 \notin K$, we choose u_2 on the affine convex segment $[u_1, \mathrm{T}u_1] \in \partial K$, such that

$$u_2 = (1-\lambda)u_1 + \lambda \mathrm{T}u_1,$$

which is equivalent to

$$d(u_1, \mathrm{T}u_1) = d(u_1, v) + d(v, \mathrm{T}u_1),$$

$v \in \partial K$.

Let $z = u_0 \in \partial K$ and $m = \frac{u_0+\mathrm{T}u_1}{2} = u_1$ the mid point of u_0 and $\mathrm{T}u_1$. By using (v) of Theorem 3.1 we get

$$\begin{aligned}\Big[d(u_0, \frac{u_0+\mathrm{T}u_1}{2})\Big]^2 &\preceq \alpha\Big[\frac{d(u_0,u_1)}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(u_0,\mathrm{T}u_1)}{\sqrt{2}}\Big]^2 \\ &\quad -(1-\alpha-\beta)\Big[\frac{d(u_1,\mathrm{T}u_1)}{2}\Big]^2,\end{aligned}$$

satisfy the Hadamard extended interpolative equation.

Repeating the above steps through induction, we can construct two sequences $\{u_n\}$ and $\{v_n\}$ whose terms satisfy the following conditions

(i) $v_n = \mathrm{T}u_n$ and $u_{n+1} = v_{n+1}$,

(ii) $u_n = \mathrm{T}u_{n-1}$, if $\mathrm{T}u_{n-1} \in K$,

(iii) $u_n = v_n$ if and only if $v_n \in K$, $u_{n+1} \neq v_{n+1}$,

(iv) $u_n \neq v_n$ whenever $v_n \notin K$ and $u_n \in \partial K$ such that u_n on the affine convex segment $[u_n, \mathrm{T}u_n] \in \partial K$ which implies that

$$u_n = (1-\lambda)u_{n-1} + \lambda \mathrm{T}u_{n-1}.$$

If $z = u_{n-1} \in \partial K$ and $m = \frac{u_{n-1}+\mathrm{T}u_n}{2} = u_n$ be the midpoint of affine convex segment $[u_n, \mathrm{T}u_n]$, we have

$$\begin{aligned}\Big[\frac{d(u_n,\mathrm{T}u_n)}{2}\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-1},u_n)}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(u_{n-1},\mathrm{T}u_n)}{\sqrt{2}}\Big]^2 \\ &\quad -(1-\alpha-\beta)\Big[d(u_{n-1}, \frac{u_{n-1}+\mathrm{T}u_n}{2})\Big]^2.\end{aligned}$$

We partition the sequence $\{u_n\}$ and $\{v_n\}$ into two sets P and Q.

$$\begin{aligned} P &= K = \{u_i \in \{u_n\} : u_i = v_i = \mathrm{T}u_{i-1}, i = 1,2,3\ldots\}, \\ Q &= \partial K = \{u_i \in \{u_n\} : u_i \neq v_i = \mathrm{T}u_{i-1}, i = 1,2,3\ldots\}.\end{aligned}$$

From this, we have three cases to investigate.

(i) Both $u_n, u_{n+1} \in P$,

(ii) If $u_n \in P$ and $u_{n+1} \in Q$,

(iii) If $u_n \in Q$ and $u_{n+1} \in P$.

Case 1 Let $u_n, u_{n+1} \in P$. Using property M, we have

$$d(u_n, u_{n+1}) \quad = \quad d(u_n, \mathrm{T}u_n) \preceq d(\mathrm{T}u_{n-1}, \mathrm{T}u_n).$$

Let $u = u_{n-1}$ and $v = u_n$ in (3.1), we get

$$\begin{aligned} d(\mathrm{T}u_{n-1}, \mathrm{T}u_n) &\preceq \vartheta[\alpha d(u_{n-1}, u_n) + \beta d(u_{n-1}, \mathrm{T}u_{n-1})] \\ &\quad +(1-\alpha-\beta) d(u_n, \mathrm{T}u_n), \\ d(u_n, u_{n+1}) &\preceq \vartheta[\alpha d(u_{n-1}, u_n) + \beta d(u_{n-1}, u_n)] + \\ &\quad (1-\alpha-\beta) d(u_n, u_{n+1}), \\ d(u_n, u_{n+1}) - (1-\alpha-\beta) d(u_n, u_{n+1}) &\preceq \vartheta[\alpha d(u_{n-1}, u_n) + \beta d(u_{n-1}, u_n)], \\ (\alpha+\beta) d(u_n, u_{n+1}) &\preceq \vartheta(\alpha+\beta) d(u_{n-1}, u_n), \\ d(u_n, u_{n+1}) &\preceq \frac{\vartheta(\alpha+\beta)}{(\alpha+\beta)} d(u_{n-1}, u_n), \\ d(u_n, u_{n+1}) &\preceq \vartheta d(u_{n-1}, u_n), \end{aligned}$$

for all $n \geq 0$.

Case 2 Given $u_n = v_n$ if and only if $v_n \in K$, $u_{n+1} \neq v_{n+1} \in \partial K$. Let $u_n \in P$ and $u_{n+1} \in Q$, we have $u_n = \mathrm{T}u_{n-1} \in P$ and $u_{n+1} \neq \mathrm{T}u_n \in Q$. (a contradiction)

By convexity relation we obtain

$$d(u_n, u_{n+1}) + d(u_{n+1}, \mathrm{T}u_n) \quad = \quad d(u_n, \mathrm{T}u_n).$$

Therefore

$$d(u_n, u_{n+1}) \quad \leq \quad d(u_n, \mathrm{T}u_n) = d(\mathrm{T}u_{n-1}, \mathrm{T}u_n).$$

Let $u = u_{n-1}$ and $v = u_n$ in (3.1), we get

$$\begin{aligned} d(\mathrm{T}u_{n-1}, \mathrm{T}u_n) &\preceq \vartheta[\alpha d(u_{n-1}, u_n) + \beta d(u_{n-1}, \mathrm{T}u_{n-1})] \\ &\quad +(1-\alpha-\beta) d(u_n, \mathrm{T}u_n), \\ d(u_n, u_{n+1}) &\preceq \vartheta[\alpha d(u_{n-1}, u_n) + \beta d(u_{n-1}, u_n)] + \\ &\quad (1-\alpha-\beta) d(u_n, u_{n+1}), \\ d(u_n, u_{n+1}) - (1-\alpha-\beta) d(u_n, u_{n+1}) &\preceq \vartheta[\alpha d(u_{n-1}, u_n) + \beta d(u_{n-1}, u_n)], \\ (\alpha+\beta) d(u_n, u_{n+1}) &\preceq \vartheta(\alpha+\beta) d(u_{n-1}, u_n), \\ d(u_n, u_{n+1}) &\preceq \frac{\vartheta(\alpha+\beta)}{(\alpha+\beta)} d(u_{n-1}, u_n), \\ d(u_n, u_{n+1}) &\preceq \vartheta d(u_{n-1}, u_n), \end{aligned}$$

for all $n \geq 0$, which is a contradiction. Hence yields an equation in case 1.

Suppose $z = u_{n-1} \in \partial K$ and $u = m = \frac{u_{n-1}+\mathrm{T}u_n}{2} = u_n \in K$ be the midpoint of affine convex segment $[u_n, \mathrm{T}u_n]$ and $v = u_{n+1} \in \partial K$, we have

$$\begin{aligned}
\Big[\frac{d(u_n,\mathrm{T}u_n)}{2}\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-1},u_n)}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(u_{n-1},\mathrm{T}u_n)}{\sqrt{2}}\Big]^2 \\
&\quad -(1-\alpha-\beta)\Big[d(u_{n-1},\frac{u_{n-1}+\mathrm{T}u_n}{2})\Big]^2, \\
\Big[\frac{d(u_n,u_{n+1})}{2}\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-1},u_n)}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(u_{n-1},u_{n+1})}{\sqrt{2}}\Big]^2 \\
&\quad -(1-\alpha-\beta)\Big[d(u_{n-1},u_n\Big]^2.
\end{aligned}$$

By $(CM3)$, we have

$$d(u_{n-1},u_{n+1}) \preceq d(u_{n-1},u_n) + d(u_n,u_{n+1}).$$

Assume that

$$\frac{(d(u_{n-1},u_n)+d(u_n,u_{n+1}))^2}{2} \preceq d(u_n,u_{n+1})^2.$$

Using the above inequalities, we get

$$\begin{aligned}
\Big[\frac{d(u_n,u_{n+1})}{2}\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-1},u_n)}{\sqrt{2}}\Big]^2 + \beta\Big[d(u_n,u_{n+1})\Big]^2 \\
&\quad -(1-\alpha-\beta)\Big[d(u_{n-1},u_n\Big]^2, \\
\Big[\frac{d(u_n,u_{n+1})}{2}\Big]^2 - \beta\Big[d(u_n,u_{n+1})\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-1},u_n)}{\sqrt{2}}\Big]^2 \\
&\quad -(1-\alpha-\beta)\Big[d(u_{n-1},u_n\Big]^2. \\
\frac{4-\beta}{4}d(u_n,u_{n+1})^2 &\preceq \frac{\alpha+2\beta-2}{2}d(u_{n-1},u_n)^2,
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
d(u_n,u_{n+1})^2 &\preceq \frac{2\alpha+4\beta-4}{4-\beta}d(u_{n-1},u_n)^2. \\
d(u_n,u_{n+1}) &\preceq \sqrt{\frac{2\alpha+4\beta-4}{4-\beta}}d(u_{n-1},u_n).
\end{aligned}$$

Letting $\vartheta = \sqrt{\frac{2\alpha+4\beta-4}{4-\beta}}$, we obtain

$$d(u_n,u_{n+1}) \preceq \vartheta d(u_{n-1},u_n).$$

Case 3 Let $u_n \neq v_n$ whenever $v_n \notin K$ and $u_{n-1}, u_n \in \partial K$ such that u_n on the affine convex segment $[u_n, \mathrm{T}u_n] \in \partial K$ which implies that

$$u_n = (1-\lambda)u_{n-1} + \lambda \mathrm{T}u_n.$$

If $u_n \in Q$ and $u_{n+1} \in P$. Then $u_{n-1} \in P$. By property M, it follows that

$$d(u_n, u_{n+1}) \quad = \quad d(u_n, \mathrm{T}u_n) \leq d(u_{n-1}, \mathrm{T}u_{n-1}) \leq d(\mathrm{T}u_{n-2}, \mathrm{T}u_{n-1}).$$

Let $u = u_{n-2}$ and $v = u_{n-1}$ in (3.1), we obtain

$$\begin{aligned} d(\mathrm{T}u_{n-2}, \mathrm{T}u_{n-1}) &\preceq \vartheta[\alpha d(u_{n-2}, u_{n-1}) + \beta d(u_{n-2}, \mathrm{T}u_{n-2})] \\ &\quad +(1-\alpha-\beta) d(u_{n-1}, \mathrm{T}u_{n-1}), \\ d(u_{n-1}, u_n) &\preceq \vartheta[\alpha d(u_{n-2}, u_{n-1}) + \beta d(u_{n-2}, u_{n-1})] + \\ &\quad (1-\alpha-\beta) d(u_{n-1}, u_n), \\ d(u_{n-1}, u_n) - (1-\alpha-\beta) d(u_{n-1}, u_n) &\preceq \vartheta[\alpha d(u_{n-2}, u_{n-1}) + \beta d(u_{n-2}, u_{n-1})], \\ (\alpha+\beta) d(u_{n-1}, u_n) &\preceq \vartheta(\alpha+\beta) d(u_{n-2}, u_{n-1}), \\ d(u_{n-1}, u_n) &\preceq \frac{\vartheta(\alpha+\beta)}{(\alpha+\beta)} d(u_{n-2}, u_{n-1}), \\ d(u_{n-1}, u_n) &\preceq \vartheta d(u_{n-2}, u_{n-1}), \\ d(\mathrm{T}u_{n-2}, \mathrm{T}u_{n-1}) &\preceq \vartheta d(u_{n-2}, u_{n-1}). \end{aligned}$$

Hence

$$d(u_n, u_{n+1}) \quad \preceq \quad d(\mathrm{T}u_{n-2}, \mathrm{T}u_{n-1}) \preceq \vartheta d(u_{n-2}, u_{n-1}).$$

Suppose $z = u_{n-2} \in \partial K$ and $u = m = \frac{u_{n-2}+\mathrm{T}u_{n-1}}{2} = u_{n-1} \in K$ be the midpoint of affine convex segment $[u_{n-1}, \mathrm{T}u_{n-1}]$ and $v = u_n \in \partial K$, we have

$$\begin{aligned} \Big[\frac{d(u_{n-1}, \mathrm{T}u_{n-1})}{2}\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-2}, u_{n-1})}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(u_{n-2}, \mathrm{T}u_{n-1})}{\sqrt{2}}\Big]^2 \\ &\quad -(1-\alpha-\beta)\Big[d(u_{n-2}, \frac{u_{n-2}+\mathrm{T}u_{n-1}}{2})\Big]^2, \\ \Big[\frac{d(u_{n-1}, u_n)}{2}\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-2}, u_{n-1})}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(u_{n-2}, u_n)}{\sqrt{2}}\Big]^2 \\ &\quad -(1-\alpha-\beta)\Big[d(u_{n-2}, u_{n-1}\Big]^2. \end{aligned}$$

By $(CM3)$, we have

$$d(u_{n-2}, u_n) \quad \preceq \quad d(u_{n-2}, u_{n-1}) + d(u_{n-1}, u_n).$$

Assume that

$$\frac{(d(u_{n-2}, u_{n-1}) + d(u_{n-1}, u_n))^2}{2} \quad \preceq \quad d(u_{n-1}, u_n)^2.$$

Using the above inequalities, we get

$$\begin{aligned}
\Big[\frac{d(u_{n-1},u_n)}{2}\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-2},u_{n-1})}{\sqrt{2}}\Big]^2+\beta\Big[d(u_{n-1},u_n)\Big]^2\\
&\quad -(1-\alpha-\beta)\Big[d(u_{n-2},u_{n-1}\Big]^2,\\
\Big[\frac{d(u_{n-1},u_n)}{2}\Big]^2-\beta\Big[d(u_{n-1},u_n)\Big]^2 &\preceq \alpha\Big[\frac{d(u_{n-2},u_{n-1})}{\sqrt{2}}\Big]^2\\
&\quad -(1-\alpha-\beta)\Big[d(u_{n-2},u_{n-1}\Big]^2.\\
\frac{4-\beta}{4}d(u_{n-1},u_n)^2 &\preceq \frac{\alpha+2\beta-2}{2}d(u_{n-2},u_{n-1})^2.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
d(u_{n-1},u_n)^2 &\preceq \frac{2\alpha+4\beta-4}{4-\beta}d(u_{n-2},u_{n-1})^2,\\
d(u_{n-1},u_n) &\preceq \sqrt{\frac{2\alpha+4\beta-4}{4-\beta}}d(u_{n-2},u_{n-1}),\\
d(\mathrm{T}u_{n-2},\mathrm{T}u_{n-1}) &\preceq \sqrt{\frac{2\alpha+4\beta-4}{4-\beta}}d(u_{n-2},u_{n-1}).
\end{aligned}$$

Letting $\vartheta=\sqrt{\frac{2\alpha+4\beta-4}{4-\beta}}$, we obtain

$$d(u_n,u_{n+1}) \preceq \vartheta d(u_{n-1},u_n).$$

By combining all three cases, we conclude that a sequence $\{u_n\}$ and $\{v_n\}$ satisfy the following inequality

$$|d(u_n,u_{n+1})| \preceq \vartheta|\max\{d(u_{n-2},u_{n-1}),d(u_{n-1},u_n)\}|.$$

For $n\geq 0$, we have

$$|d(u_n,u_{n+1})| \preceq \vartheta^{\frac{n-1}{2}}|\max\{d(u_{n-2},u_{n-1}),d(u_{n-1},u_n)\}|.$$

By induction, we establish the cases $n=0,n=1,n=2,n=3$.

For $n=0$, we have

$$|d(u_0,u_1)| \preceq \vartheta^{\frac{-1}{2}}|\max\{d(u_{-2},u_{-1}),d(u_{-1},u_0)\}|.$$

For $n=1$, we have

$$|d(u_1,u_2)| \preceq |\max\{d(u_{-1},u_0),d(u_0,u_1)\}|.$$

For $n=2$, we have

$$|d(u_2,u_3)| \preceq \vartheta^{\frac{1}{2}}|\max\{d(u_0,u_1),d(u_1,u_2)\}|.$$

For $n = 3$, we have

$$|d(u_3, u_4)| \preceq \vartheta^1 |\max\{d(u_1, u_2), d(u_2, u_3)\}|.$$

Letting $\sigma_n = \max\{d(u_{n-2}, u_{n-1}), d(u_{n-1}, u_n)\}$, for $n \geq 0$ we have

$$\lim_{n\to\infty} |d(u_n, u_{n+1})| = \vartheta^{\frac{n-1}{2}} \sigma_n.$$

For $n \to \infty$ the sequence $\{u_n\}$ converges to 0, using Lemma 2.2 and Definition 2.5 implies that

$$\lim_{n\to\infty} |d(u_n, u_{n+1})| \to 0 \ as \ n \to \infty.$$

We show that $\{u_n\}$ is a Cauchy sequence. From Definition 2.5, Lemma 2.3 and $(CM3)$, for $n, m \in \mathbb{N}$ such that $n > m$, we have

$$\begin{aligned} d(u_n, u_m) &\preceq d(u_n, u_{n-1}) + d(u_{n-1},, u_{n-2}) + d(u_{n-2},, u_{n-3}) + \cdots + \\ &\quad d(u_{m+1}, u_m) + d(u_m, u_{m-1}), \\ &\preceq [\vartheta^{\frac{n-1}{2}} + \vartheta^{\frac{n-2}{2}} + \cdots + \vartheta^{\frac{m-1}{2}}]\sigma_n, \\ &\preceq \frac{(\vartheta^{\frac{1}{2}})^{n-1}}{1 - \vartheta^{\frac{1}{2}}} \sigma_n, \\ &\preceq \frac{(\sqrt{\vartheta})^{n-1}}{1 - \sqrt{\vartheta}} \sigma_n \to \ 0 \ as \ n \to \ \infty. \end{aligned}$$

Hence

$$|d(u_n, u_{n+1})| \to 0,$$

which implies that $(u_n, u_{n+1}) = 0$. Therefore $\{u_n\}$ for all $n \in \mathbb{N}$ is a Cauchy sequence in K.

For completeness of K, let $u^\star \in K$ such that

$$\lim_{n\to\infty} u_n = u^\star$$

and

$$z \in K, \mathrm{T}u^\star = z.$$

Since T is sequentially convergent in K, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $\{v_{n_k}\}$ of $\{v_n\}$ such that

$$\lim_{n\to\infty} u_{n_k} = v_{n_k} = \mathrm{T}u_{n_{k-1}}.$$

Implies that

$$\lim_{n\to\infty} \mathrm{T}u_{n_{k-1}} = z.$$

Now we show that $\mathrm{T}u^\star = z$, using $(CM3)$, we have

$$d(\mathrm{T}u^\star, z) \preceq d(\mathrm{T}u^\star, \mathrm{T}u_{n_{k-1}}) + d(\mathrm{T}u_{n_{k-1}}, z).$$

Let $u = u^\star$ and $v = u_{n_{k-1}}$ using (3.1), we obtain

$$\begin{aligned} d(\mathrm{T}u^\star, \mathrm{T}u_{n_{k-1}}) &\preceq \vartheta[\alpha d(u^\star, u_{n_{k-1}}) + \beta d(u^\star, \mathrm{T}u^\star)] \\ &\quad +(1-\alpha-\beta) d(u_{n_{k-1}}, \mathrm{T}u_{n_{k-1}}), \end{aligned}$$

$$\begin{aligned} d(\mathrm{T}u^\star, z) &\preceq \vartheta[\alpha d(u^\star, u_{n_{k-1}}) + \beta d(u^\star, \mathrm{T}u^\star)] \\ &\quad +(1-\alpha-\beta) d(u_{n_{k-1}}, \mathrm{T}u_{n_{k-1}}) + d(\mathrm{T}u_{n_{k-1}}, z). \\ &\preceq \vartheta[\alpha d(u^\star, z) + \beta d(u^\star, z)] + (1-\alpha-\beta) d(z, z) + d(z, z), \\ &\preceq \vartheta(\alpha+\beta) d(u^\star, z), \\ d(\mathrm{T}u^\star, z) &\preceq 0. \end{aligned}$$

This implies that $d(\mathrm{T}u^\star, z) = 0$. Thus, $\mathrm{T}u^\star = z$, hence z is a fixed point of T in K.

The uniqueness of z in K, we assume that $w \in K$ is another fixed point of T in K such that $u = z$ and $v = w$, using using (3.1) we get

$$\begin{aligned} d(\mathrm{T}z, \mathrm{T}w) &\preceq \vartheta[\alpha d(z, w) + \beta d(z, \mathrm{T}z)] + (1-\alpha-\beta) d(w, \mathrm{T}w), \\ d(z, w) &\preceq \vartheta[\alpha d(z, w) + \beta d(z, z)] + (1-\alpha-\beta) d(w, w), \\ d(z, w) &\preceq \vartheta\alpha d(z, w), \\ d(z, w) - \vartheta\alpha d(z, w) &\preceq 0, \\ (1-\vartheta\alpha) d(z, w) &\preceq 0, \\ d(z, w) &\preceq 0, \\ d(z, w) &= 0. \end{aligned}$$

Implies that $z = w$. Thus z is a unique fixed point of T in K. Q.E.D.

The corollary provided below is the extended interpolative rational type mapping from [12] using hyperbolic complex metric space for the novelty of Theorem 3.1.

Corollary 3.1. Let (X, d) be a hyperbolic complex-valued metric space, K a non-empty closed subset of X, and ∂K the boundary of K. Let ∂K be non empty and $u_0 \in K$. Let $\mathrm{T} : K \to X$ be an extended interpolative Gupta-type contraction non-self mapping, then the following condition holds:

(i) $\mathrm{T}(\partial K) \subset K$,

(ii) T has property M,

(iii) T is sequentially convergent in K,

(iv)

$$d(\mathrm{T}u, \mathrm{T}v) \preceq \vartheta_1 d(u, v) + (1-\vartheta_1) \frac{d(u, \mathrm{T}u) d(v, \mathrm{T}v)}{d(u, v)},$$

for all $u, v \in K$, where $\vartheta_1 < 1$.

(v) if $z \in$ affine convex segment $[u, v]$ and $m = \frac{u+v}{2}$ the midpoint of $[u, v]$, then the Hadamard extended interpolative equation

$$\Big[\frac{d(u,v)}{2}\Big]^2 \preceq \alpha\Big[\frac{d(z,u)}{\sqrt{2}}\Big]^2 + \beta\Big[\frac{d(z,v)}{\sqrt{2}}\Big]^2 - (1-\alpha-\beta)\Big[d(z,m)\Big]^2,$$

is satisfied with $\triangle(z, u, v)$ a geodesic triangle in K and $0 < \vartheta \leq 1$, $\vartheta = \sqrt{\frac{2\alpha+4\beta-4}{4-\beta}}$.

Then, there exists a unique fixed point of T in K.

Proof. The proof of this corollary follow the similar proof of Theorem 3.1. This completes the proof. Q.E.D.

We formulate an example for verification of the results proved above.

Example 3.1. Let $X = \partial K = [0, 1], K = \Big\{0, \frac{1}{2}, \frac{1}{4}\Big\}$. Denote the unit interval of real numbers and $\mathrm{T} : K \to X$ given by

$$\mathrm{T}u = \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)u^n,$$

for $u, v \in K$.

Let $X = [0, 1)$ and $d : K \times X \to \mathbb{C}$ be defined by

$$d(u,v) = |u-v|e^{ik},$$

where $k \in [0, \frac{\pi}{2}]$ and $e^{ik} = \cos k + i \sin k$. Then, (X, d) is a complete hyperbolic complex-valued metric space.

To demonstrate the conditions in Theorem 3.1, we calculate the following hyperbolic complex-valued distances.

$$\begin{aligned} d(\mathrm{T}u, \mathrm{T}v) &= d\Big(\Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)u^n, \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)v^n\Big) \\ &= \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)\Big|u^n - v^n\Big|\Big(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\Big), \\ &= \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)\Big|u^n - v^n\Big|i, \end{aligned}$$

$$\begin{aligned} d(u, \mathrm{T}u) &= d\Big(u, \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)u^n\Big) \\ &= \Big|\frac{2^{n+1}(u-u^n)+u^n}{2^{n+1}}\Big|\Big(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\Big), \\ &= \Big|\frac{2^{n+1}(u-u^n)+u^n}{2^{n+1}}\Big|i, \end{aligned}$$

$$\begin{aligned} d(v, \mathrm{T}v) &= d\Big(v, \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)v^n\Big) \\ &= \Big|\frac{2^{n+1}(v-v^n)+v^n}{2^{n+1}}\Big|\Big(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\Big), \\ &= \Big|\frac{2^{n+1}(v-v^n)+v^n}{2^{n+1}}\Big|i, \end{aligned}$$

$$\begin{aligned} d(u,v) &= |u-v|e^{ik}, \\ d(u,v) &= |u-v|\Big(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\Big), \\ d(u,v) &= |u-v|i. \end{aligned}$$

Applying all of the equalities above in (3.1), we get

$$\begin{aligned} \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)\Big|u^n-v^n\Big|i &\preceq \vartheta\Big[\alpha|u-v|i+\beta\Big|\frac{2^{n+1}(u-u^n)+u^n}{2^{n+1}}\Big|i\Big] \\ &\quad +(1-\alpha-\beta)\Big|\frac{2^{n+1}(v-v^n)+v^n}{2^{n+1}}\Big|i. \end{aligned}$$

Letting $\alpha = \frac{1}{2}, \beta = \frac{1}{4}$ and $\vartheta = \frac{1}{3}$ in the above inequality, we obtain

$$\begin{aligned} \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)\Big|u^n-v^n\Big|i &\preceq \frac{1}{3}\Big[\frac{1}{2}|u-v|i+\frac{1}{4}\Big|\frac{2^{n+1}(u-u^n)+u^n}{2^{n+1}}\Big|i\Big] \\ &\quad +\frac{1}{4}\Big|\frac{2^{n+1}(v-v^n)+v^n}{2^{n+1}}\Big|i. \end{aligned}$$

Equivalently to

$$\begin{aligned} \Big(\frac{2^{n+1}-1}{2^{n+1}}\Big)\Big|u^n-v^n\Big| &\preceq \frac{1}{3}\Big[\frac{1}{2}|u-v|+\frac{1}{4}\Big|\frac{2^{n+1}(u-u^n)+u^n}{2^{n+1}}\Big|\Big] \\ &\quad +\frac{1}{4}\Big|\frac{2^{n+1}(v-v^n)+v^n}{2^{n+1}}\Big|. \end{aligned}$$

Hence, for all $n \geq 0$ the hypothesis of Theorem 3.1 are satisfied. Therefore a mappings T has a unique fixed point in K.

4 An application of Hadamard partial fractional differential equation in hyperbolic complex-valued metric spaces

In this section, we aim to investigate the uniqueness of the solution of Hadamard partial fractional differential equation for demonstration of Theorem 3.1. Butzer *et al.* [8] investigated Fractional calculus in the Mellin setting and Hadamard-type fractional integrals and their derivatives. Butzer *et al.* [9], they obtained the Mellin transform analysis and integration by parts for Hadamard-type fractional integrals and differential operators. Pooseh *et al.* [28] obtained expansion formulas in terms of integral order derivatives for the Hadamard fractional integral and derivative. Samko *et al.* [33] derived more operators on fractional Integrals and Derivatives.

Abbas *et al.* [1] considered the uniqueness of the solution to the following Hadamard partial fractional integral equation

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st} dt\ ds, \tag{4.1}$$

for all $t, s \in K = [1,a] \times [1,b], a, b > 1, r_1, r_2 > 0$ where $K : \mathcal{X} \times \mathcal{X} \times \mathbb{R}^n \to \mathbb{C}$ and $\mu : K \to \mathbb{R}, f : K \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $X = C(X, \mathbb{R}^n)$ on $\mathbb{C}$ be Banach space of continuous functions $\mu : K \to \mathbb{R}$ with the norm

$$\|u\|_X = \sup_{(x,y)\in K} |u|,$$

and $L^{'}(K,\mathbb{R})$ be Banach space of continuous $\mu : K \to \mathbb{R}$ that are Lebesgue integral with norm

$$\|u\|_{L'} = \int_1^a \int_1^b |u(x,y)| dy\ dx.$$

Definition 4.1. [15, 18] The Hadamard fractional integral order $q > 0$ for a function $g \in L^{'}([1,a],\mathbb{R})$, is defined by

$$({}^H I_1^r g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\log \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 4.2. [15, 18] Let $r_1, r_2 \geq 0, \sigma(1,1)$ and $r = (r_1, r_2)$ for $v \in L^{'}(K,\mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression function $g \in L^{'}([1,a],\mathbb{R})$, is defined by

$$({}^H I_\sigma^r v)(x,y) = \frac{1}{\Gamma(r_1\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{v(s,t)}{st} dtds,$$

Define a hyperbolic complex-valued metric on X, by $d : X \times X \to \mathbb{C}$ and

$$d(u,v) = \max_{(x,y)\in K} |u-v| e^{ik}, k \in [0, \frac{\pi}{2}].$$

Then (X,d) is a hyperbolic complete complex-value metric space.

Theorem 4.1. Suppose that the following conditions holds:

(i) $f : K \times \mathbb{R} \to \mathbb{R}$ and $\mu : \mathbb{R} \to \mathbb{R}$ are continuous functions,

(ii) there exists $f : K \times \mathbb{R} \to \mathbb{R}$ such that

$$\|f(s,t,u(s,t)) - f(s,t,v(s,t))\| \preceq \vartheta \|u-v\| e^{ik},$$

where

$$d(u,v) = \max_{(x,y)\in K} |u-v|e^{ik}, k \in [0, \frac{\pi}{2}].$$

and

$$d(u,v) = d(\mathrm{T}u, \mathrm{T}v) \preceq \vartheta[\alpha d(u,v) + \beta d(u, \mathrm{T}u)] + (1-\alpha-\beta)d(v, \mathrm{T}v),$$

for all $u, v \in K$.

(iii) there exists $\vartheta \in [0,1)$ such that

$$\frac{4(r_1-1)(r_2-1)}{\Gamma(r_1)\Gamma(r_2)r_1r_2} \ln x \ln y \preceq \vartheta, \tag{4.2}$$

for $r_1, r_2 > 0, \vartheta < 1$ and $(x,y) \in K$.

Then, the partial fractional differential equation (4.1) has a unique solution in K.

Proof. Let us define a non-self-mappings $\mathrm{T} : K \to X$ by

$$\mathrm{T}u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st} dt\ ds \tag{4.3}$$

for all $t, s \in K$.

Let $u, v \in C([1,1],\mathbb{R})$ with $u \prec v$, we assume that $d(\mathrm{T}u, \mathrm{T}v) \geq \vartheta d(u,v))$. By the conditions $(i), (ii) and (iii)$, we have

$$\begin{aligned} \|\mathrm{T}u(x,y) - \mathrm{T}v(x,y)\| \preceq & \max_{(x,y)\in K} \Big\| \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \\ & \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st} dt\ ds \Big\| - \\ & \max_{(x,y)\in K} \Big\| \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \\ & \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,v(s,t))}{st} dt\ ds \Big\|. \end{aligned} \tag{4.4}$$

Through integration by parts we obtain the following

$$\int_1^y \left(\log\frac{y}{t}\right)^{r_1-1} . \frac{1}{t} dt = \frac{2(r_2-1)}{r_2} \ln y, \tag{4.5}$$

and

$$\int_1^x \left(\log\frac{x}{s}\right)^{r_1-1} . \frac{1}{s} ds = \frac{2(r_1-1)}{r_1} \ln x. \tag{4.6}$$

Using (4.5) and (4.6) in (4.4), we obtain

$$\begin{aligned}
\|\mathrm{T}u(x,y)-\mathrm{T}v(x,y)\| &\preceq \max_{(x,y)\in K}\bigg\|\frac{1}{\Gamma(r_1)\Gamma(r_2)}.\frac{2(r_1-1)}{r_1}\ln x.\frac{2(r_2-1)}{r_2}\ln y\\
&\quad f(s,t,u(s,t))-f(s,t,v(s,t)\bigg\|,\\
&\preceq \max_{(x,y)\in K}\bigg\|\frac{2(r_1-1)2(r_2-1)}{\Gamma(r_1)\Gamma(r_2)r_1r_2}.\ln x.\ln y\\
&\quad f(s,t,u(s,t))-f(s,t,v(s,t)\bigg\|,\\
&\preceq \frac{2(r_1-1)2(r_2-1)}{\Gamma(r_1)\Gamma(r_2)r_1r_2}.\ln x.\ln y\\
&\quad \max_{(x,y)\in K}\|f(s,t,u(s,t))-f(s,t,v(s,t),\|\\
\|\mathrm{T}u(x,y)-\mathrm{T}v(x,y)\| &\preceq \vartheta\|u-v\|e^{ik}.
\end{aligned}$$

Equivalently to

$$d(\mathrm{T}u,\mathrm{T}v) \preceq \vartheta d(u,v),$$

Implies that

$$d(\mathrm{T}u,\mathrm{T}v) \preceq \vartheta[\alpha d(u,v)+\beta d(u,\mathrm{T}u)]+(1-\alpha-\beta)d(v,\mathrm{T}v),$$

which is a contradiction to the claim that $d(\mathrm{T}u,\mathrm{T}v)\geq\vartheta d(u,v))$. Therefore a mapping T has a unique fixed point u in K, which is a solution of a partial fractional differential equation (4.1). Thus, this verifies the conditions imposed in Theorem 3.1 and Theorem 4.1. Hence, the proof is completed. Q.E.D.

Acknowledgements.
Authors would like to thank the MUST administration for this research to be conducted at the University.

Bibliography

[1] Abbas, S., Benchohra, M. and Henderson, J., 2015. Partial Hadamard fractional integral equations. *Adv. Dyn. Syst. Appl,* 10 (2), 97-107.

[2] Assad, N. A. and Kirk, W. A., 1972. Fixed point theorems for set-valued mappings of contractive type. *Pacific Journal of Mathematics*, 43(3), 553–562.

[3] Azam, A., Fisher, B. and Khan, M., 2011. Common fixed point theorems in complex valued metric spaces. *Numerical Functional Analysis and Optimization,* 32(3), 243-253.

[4] Berinde, V. and Pacurar, M., 2013. Fixed point theorems for nonself single-valued almost contractions. *Fixed Point Theory,*, 14(2):301-312, 2013.

[5] Boyd, D. W. and Wong, J. S, 1969. On nonlinear contractions. *Proceedings of the American Mathematical Society,* **20**(2), 458–464.

[6] Branciari, A., 2000. A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces. *Publ. Math. Debrecen,* 57(2000), 31-37.

[7] Bruhat, F. and Tits, J., 1972. Groupes réductifs sur un corps local. I. Données radicielles valuées. *Inst. Hautes Études Sci. Publ. Math,* 41, 5–251.

[8] Butzer, P. L., Kilbas, A. A. and Trujillo. J. J., 2002. Fractional calculus in the Mellin setting and Hadamard-type fractional integrals. *J. Math. Anal. Appl.* 269(1), 1-27.

[9] Butzer, P. L., Kilbas, A. A. and Trujillo. J. J., 2002. Mellin transform analysis and integration by parts for Hadamard-type fractional integrals. *J. Math. Anal. Appl.* 270(1), 1-15.

[10] Camargo, J.L.C., 1988. An application of a fixed point theorem of D.W. Boyd and J.S.W. Wong. *Rev. Mat. Estatist.,* 6, 25–29.

[11] Ćirić, L., Rakočević, V., Radenović, S., Rajović, M. and Lazović, R., 2010. Common fixed point theorems for non-self-mappings in metric spaces of hyperbolic type. *J. Comp. Appl. Math.,* 233(11), 2966-2974.

[12] Dass, B.K., Gupta, S., 1975. An extension of Banach contraction principle through rational expression.*India Journal of Pure Appl. Math.,* 6(12), 1455-1458 (1975) https://doi.org/10.1155/2020/7816505.

[13] Eke, K.S. and Oghonyon, J.G., 2018. Common fixed point theorems for non-self mappings of nonlinear contractive maps in convex metric spaces. *Journal of Mathematics and Computer Science.,* 6(18), 181-191.

[14] Gautam, P., Mishra, V.N., Ali, R. and Verma, S., 2021. Interpolative Chatterjea and cyclic Chatterjea contraction on quasi-partial b-metric space. *AIMS Mathematics,* 6(2), 1727-1742.

[15] Hadamard, J., 1892. Essai sur l'etude des fonctions donn'ees par leur d'eveloppment de Taylor. *J. Pure Appl. Math.* 4 (8), 101–186.

[16] Imdad, M. and Kumar, S., 2003. Rhoades-type fixed-point theorems for a pair of nonself mappings. *Comp. Math. Appl.,* 46(5-6), 919–927.

[17] Karapinar, E., 2018. Revisiting the Kannan type contractions via interpolation. *Adv. Theory Nonlinear Anal. Appl.,* 2(2), 85–87.

[18] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., 2006. Theory and Applications of Fractional Differential Equations, *Elsevier Science B.V., Amsterdam.*

[19] Kirk, W. A., 1982. Krasnoselskii's iteration process in hyperbolic space. *Numer. Funct. Anal. Optim.,* 4 (4), 371-381.

[20] Klin-eam, C. and Suanoom, C., 2013. Some common fixed point theorems for generalized contractive type mappings on complex-valued metric spaces. *Abst. Appl. Anal.,* (2013), Article ID 604215.

[21] Kohlenbach, U., 2005. Some logical metatheorems with applications in functional analysis. *Trans. Am. Math,Soc.,* 357(1), 89-128.

[22] Krein, S.G., Petunin, J.I. and Semenov, E.M., 1978. Interpolation of linear operators. *Amer. Math. Soc. Prov.,* Providence, RI, USA.

[23] Kumam, P., Sarwar, M. and Zada, M.B., 2016. Fixed point results satisfying rational type contractive conditions in complex-valued metric spaces. *Ann. Math. Sil.,* 30(1), 89-110.

[24] Mishra, V.N., Sánchez Ruiz, L.M., Gautam, P. and Verma, S., 2020. Interpolative Reich–Rus–Ćirić and Hardy–Rogers contraction on quasi-partial b-metric space and related fixed point results. *Mathematics,* 8(9), 1598.

[25] Mohammadi, B., Parvaneh, V. and Aydi, H., 2019. On extended interpolative Ćirić-Reich-Rus type F-contractions and an application. *J. Ineq. Appl.,* (2019), 1-11.

[26] Moosaei, M.,2012. Fixed point theorems in convex metric spaces. *Fixed Point Theory and Applications,* 1(2012), 1–6.

[27] Papadopoulos, A., 2005. Metric Spaces, Convexity and Nonpositive Curvature. *Eur. Math. Soc.,* Vol.6.

[28] Pooseh, S., Almeida, R. and Torres, D., 2012. Expansion formulas in terms of integral order derivatives for the Hadamard fractional integral and derivative. *Numer. Funct. Anal. Optim.* 33 (3), 301–319.

[29] Reich, S. and Salinas, Z., 2016. Weak convergence of infinite products of operators in Hadamard spaces. *Rendiconti del Circolo Matematico di Palermo,* 65 (2016), 55-71.

[30] Reich, S. and Shafrir, I., 1990. Nonexpansive iterations in hyperbolic spaces. *Nonlinear Analysis: Theory, Methods and Applications,* 15(6), 537-558.

[31] Rhoades,B., 1978. A fixed point theorem for some non-self mappings. *Math. Japonica,* 23(4),457–459.

[32] Rouzkard, F. and Imdad, M.,2012. Some common fixed point theorems on complex-valued metric spaces. *Comput. Math. Appl.* 64(6), 1866-1874.

[33] Samko, S. G., Kilbas, A. A. and Marichev, O. I., 1993. Fractional Integrals and Derivatives. *Gordon and breach science publishers,* Vol. 1.

[34] Shafrir, I., 1990. The approximate fixed point property in Banach and hyperbolic spaces.*Israel J. Math.,* 71(2), 211–223.

[35] Singh, N., Singh, D., Badal, A. and Joshi, V., 2016. Fixed point theorems in complex valued metric spaces. *Journal of the Egyptian Mathematical Society,* 24(3), 402-409.

[36] Sintunavarat, W., Kumam, P.: Generalized common fixed point theorems in complex-valued metric spaces and applications. *J. Inequal. Appl.,* (2012), 1-12.

[37] Sintunavarat, W., Cho, Y.J., Kumam, P., 2013. Urysohn integral equations approach by common fixed points in complex-valued metric spaces. *Adv. Differ. Equ.,* 2013(1), 1-14.

[38] Takahashi, W., 1970. A convexity in metric space and nonexpansive mappings, *I. Kodai Math. Sem. Rep.,* 22(2), (1970), 142–149.

[39] Wangwe, L., 2022. Fixed point theorems for interpolative Kanann contraction mappings in Busemann space with an application to a matrix equation. *The Journal of Analysis,* (2022), 1-20.

[40] Wangwe, L., 2022. Fixed point theorem for interpolative mappings in F-Mv-metric space with an application. *Topological Algebra and its Applications,* 10(1), 141-153.

[41] Wangwe, L. and Kumar, S., 2022. Fixed point results for interpolativeΨ-Hardy-Rogers type contraction mappings in quasi-partial b-metric space with applications. *The Journal of Analysis,* (2022), 1-18.

[42] Wangwe, L. and Kumar, S., 2021. Fixed point theorem for multivalued non-self mappings in partial symmetric spaces, *Topological Algebra and its Applications,,* 9(1), 20-36.